Stress-Energy tensor for Maxwell Theory

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Maxwell's theory of electromagnetism can be expressed in terms of a 4-vector field A_μ, coupled to a current j^μ due to "matter" fields. The Lagrangian density given by

$$\mathcal{L} = \mathcal{L}_{\text{Max}} + \mathcal{L}_{\text{Matter}} + \mathcal{L}_{\text{Int}},$$

where

$$\mathcal{L}_{\text{Max}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad \text{and} \quad \mathcal{L}_{\text{Int}} = -A_\mu j^\mu.$$

Here the field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where, as we have seen in Homework 1, Problem 3, consistency requires $\partial_\nu j^\nu = 0$.

Consider first the stress energy tensor for the pure Maxwell theory (without a source j^μ). The action is invariant under a translation:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + c^\nu \partial_\nu A_\mu(x),$$

which means that there are conserved currents. Just as for Homework 2, Problem 3, the current is given by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\rho} \partial_\nu A_\rho c^\nu - \mathcal{L} c^\mu + c_\nu \Lambda^{\nu\mu},$$

where I have used $\delta A^\sigma = 0$ and replaced $d\delta x^\nu / d\epsilon$ with c^ν, but have not specified $\Lambda^{\nu\mu}$, which is only constrained to have $\partial_\mu \Lambda^{\nu\mu} = \delta \mathcal{L} = 0$. Previously we chose to take $\Lambda = 0$, but we will see that there is a better choice here. From Homework 1, Problem 2, we have

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\rho} = F^{\rho\mu},$$

so

$$J^\mu = F^{\rho\mu} \partial_\nu A_\rho c^\nu + \frac{1}{4} c^\mu F^{\rho\sigma} F_{\rho\sigma} + c_\nu \Lambda^{\nu\mu} =: c^\nu T^{\nu\mu},$$

or

$$T^{\nu\mu} = F^{\rho\mu}\partial^\nu A_\rho + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + \Lambda^{\nu\mu}.$$

While this $T^{\nu\mu}$ is a correct conserved current even if we drop the Λ term, there are two unpleasant features. First, it is not symmetric under $\mu \leftrightarrow \nu$, which we expect of the energy momentum tensor, is required for the angular momentum current $\mathcal{M}^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$ to be conserved, and to couple to the curvature in general relativity. Secondly, this T would not be invariant under a gauge transformation $A_\rho \to A_\rho + \partial_\rho\lambda$, so it depends on unphysical degrees of freedom.

We can remedy this problem by choosing

$$\Lambda^{\nu\mu} = -F^{\rho\mu}\partial_\rho A^\nu,$$

which satisfies

$$\partial_\mu\Lambda^{\nu\mu} = -\left(\partial_\mu F^{\rho\mu}\right) - F^{\rho\mu}\partial_\rho\partial_\mu A^\nu = 0,$$

the first term from the equation of motion and the second from the antisymmetry of $F^{\rho\mu}$ dotted into the symmetric $\partial_\rho\partial_\mu$. This term completes the F in T, so

$$T^{\nu\mu} = -F^{\rho\mu}F_\rho{}^\nu + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}.$$

To reexpress this in more familiar terms, with $F^{j0} = E^j$ and $F^{ij} = -\epsilon_{ijk}B^k$, we note first that $F^{\rho\sigma}F_{\rho\sigma} = -2(F^{0j})^2 + (F^{ij})^2 = -2\vec{E}^{\,2} + 2\vec{B}^{\,2}$, and then we have have

$$\begin{aligned}
T^{00} &= (F^{i0})^2 + \frac{1}{4}(-2\vec{E}^{\,2} + 2\vec{B}^{\,2}) \\
&= \frac{1}{2}\vec{E}^{\,2} + \frac{1}{2}\vec{B}^{\,2}, \\
T^{0i} = T^{i0} &= -F^{j0}F_j{}^i = -F^{j0}F^{ij} = E^j\epsilon_{ijk}B^k = \left(\vec{E}\times\vec{B}\right)_i, \\
T^{ij} &= F^{kj}F^{ki} - F^{0j}F^{0i} - \frac{1}{4}\delta^{ij}(-2\vec{E}^{\,2} + 2\vec{B}^{\,2}) \\
&= \epsilon_{kj\ell}B^\ell\epsilon_{kim}B^m - E^iE^j + \frac{1}{2}\delta^{ij}(\vec{E}^{\,2} - \vec{B}^{\,2}) \\
&= \frac{1}{2}\delta^{ij}(\vec{E}^{\,2} + \vec{B}^{\,2}) - B^iB^j - E^iE^j.
\end{aligned}$$