$\Gamma(N/2)$ and the Volume of S^{D-1}

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Here we evaluate the "area" of the surface of a ball of radius 1 in D dimensions, that is, the (hyper) volume of a $D-1$ dimensional sphere S^{D-1}. To do so we also need to evaluate the Euler Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{for} \quad \text{Re } z > 0.$$

Note that $\Gamma(1) = 1$ and

$$\begin{aligned} \Gamma(z+1) &= -\int_0^\infty t^z d\left(e^{-t}\right) \\ &= t^z\, e^{-t}\Big|_0^\infty + z\int_0^\infty t^{z-1} e^{-t} dt \\ &= z\Gamma(z) \end{aligned}$$

for $\text{Re } z > 0$. We can evaluate Γ for half-integer arguments and simultaneously the volume of a $D-1$ sphere by evaluating this integral in D dimensional Euclidean space:

$$I = \int_{\mathbb{R}^D} e^{-\vec{r}^2}.$$

If we do the integral using cartesian coordinates,

$$\begin{aligned} I &= \prod_{i=1}^{D} \left(\int_{-\infty}^{\infty} dr_i\right) e^{-\sum r_i^2} \\ &= \prod_{i=1}^{D} \left(\int_{-\infty}^{\infty} dr_i\, e^{-r_i^2}\right) \\ &= \left(\int_{-\infty}^{\infty} du\, e^{-u^2}\right)^D. \end{aligned}$$

The integral in the last expression is

$$\int_{-\infty}^{\infty} du\, e^{-u^2} = 2\int_0^\infty du\, e^{-u^2} \underset{t=u^2}{\longrightarrow} \int_0^\infty dt\, t^{-\frac{1}{2}} e^{-t} = \Gamma(\frac{1}{2}),$$

so $I = \left(\Gamma(\frac{1}{2})\right)^D$.

On the other hand $e^{-\vec{r}^2}$ is hyperspherically symmetric, so

$$I = S_D \int_0^\infty r^{D-1} e^{-r^2} dr = \frac{1}{2} S_D \int_0^\infty t^{\frac{D}{2}-1} e^{-t} dt = \frac{1}{2} S_D \Gamma(D/2),$$

where $S_D = \int d\Omega_D$ is the surface area of a unit ball in D dimensions. Thus we have

$$\frac{1}{2} S_D \Gamma(D/2) = \left(\Gamma(\frac{1}{2})\right)^D.$$

For $D = 2$ we know, of course, that the surface of a 2-ball, that is a circle of radius 1, has "volume" 2π, so[1]

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

Thus for all D we have

$$S_D = 2 \frac{\pi^{D/2}}{\Gamma(D/2)}.$$

From $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 1$, and from the recursion relation $\Gamma(z+1) = z\Gamma(z)$, we can evaluate

$$\Gamma(D/2) = \sqrt{\pi}, 1, \frac{\sqrt{\pi}}{2}, 1, \frac{3\sqrt{\pi}}{4}, 2, \ldots \quad \text{for } D = 1, 2, \ldots,$$

and thus

$$S_D = 2, 2\pi, 4\pi, 2\pi^2, \frac{8\pi^2}{3}, \ldots \quad \text{for } D = 1, 2, \ldots .$$

[1] Not $-\sqrt{\pi}$ as the integrand is clearly positive definite.