Notes on Bessel Functions

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Bessel functions $J_m(x)$ of integral order m may be defined by the generating function

$$g(x,t) := e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \tag{1}$$

As the generating function is unchanged by $x \to -x, t \to 1/t$, we have $J_{-n}(-x) = J_n(x)$, but it is also unchanged by $x \to -x, t \to -t$, so $J_n(-x) = (-1)^n J_n(x)$, so $J_{-n}(x) = (-1)^n J_n(x)$

Differentiating (1) with respect to t gives

$$\frac{1}{2} x \left(1 + \frac{1}{t^2}\right) g(x,t) = \sum_{\mathbb{Z}} n J_n(x) t^{n-1} = \frac{x}{2} \sum_{\mathbb{Z}} J_m \left(t^m + t^{m+2}\right)$$

$$\Longrightarrow J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x). \tag{2}$$

Differentiating (1) with respect to x gives

$$\frac{1}{2} \left(t - \frac{1}{t}\right) g(x,t) = \sum_{\mathbb{Z}} J_n'(x) t^n = \sum_{\mathbb{Z}} J_m(x) \left(t^{m+1} - t^{m-1}\right)$$

$$\Longrightarrow J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x). \tag{3}$$

As a special case, $J_0'(x) = -J_1(x)$. (2) $\mp$ (3) gives

$$J_{n\pm1}(x) = \frac{n}{x} J_n(x) \mp J_n'(x). \tag{4}$$

Manipulating Eqs. (4), even if m is not an integer, gives the Bessel equation

$$x^2 \frac{d^2}{dx^2} J_\nu(x) + x \frac{d}{dx} J_\nu(x) + (x^2 - \nu^2) J_\nu(x) = 0 \tag{5}$$

This can also be written

$$u \frac{d}{du} u \frac{d}{du} J_\nu(\alpha u) = (\nu^2 - \alpha^2 u^2) J_\nu(\alpha u)$$

Orthogonality:
Then

$$J_\nu(\alpha u) \frac{d}{du} u \frac{d}{du} J_\nu(\beta u) - J_\nu(\beta u) \frac{d}{du} u \frac{d}{du} J_\nu(\alpha u) = \left(\alpha^2 - \beta^2\right) u J_\nu(\alpha u) J_\nu(\beta u)$$

Integrate from 0 to 1:

$$\begin{aligned}
\left(\alpha^2 - \beta^2\right) \int_0^1 du\, u\, J_\nu(\alpha u) J_\nu(\beta u) &= \int_0^1 J_\nu(\alpha u) \frac{d}{du} u \frac{d}{du} J_\nu(\beta u) - (\alpha \leftrightarrow \beta) \\
&= \left[u J_\nu(\alpha u) \frac{d}{du} J_\nu(\beta u) \bigg|_0^1 - \int_0^1 u \left(\frac{d}{du} J_\nu(\alpha u)\right) \frac{d}{du} J_\nu(\beta u) \right] - (\alpha \leftrightarrow \beta) \\
&= u J_\nu(\alpha u) \frac{d}{du} J_\nu(\beta u) \bigg|_0^1 - u J_\nu(\beta u) \frac{d}{du} J_\nu(\alpha u) \bigg|_0^1 . \qquad (6)
\end{aligned}$$

For $\nu \geq 0$ we may assume $J_\nu(0)$ is finite, so the lower endpoint gives zero. If α and β are both zeros of J_ν or both zeros of J_ν', the upper endpoint also vanishes, so

$$\left.\begin{aligned}
\int_0^1 du\, u\, J_\nu(x_{\nu n} u) J_\nu(x_{\nu m} u) = 0 \\
\int_0^1 du\, u\, J_\nu(x'_{\nu n} u) J_\nu(x'_{\nu m} u) = 0
\end{aligned}\right\} \quad \text{for } m \neq n, \nu \geq 0$$

If we first differentiate (6) with respect to α, and then set $\beta = \alpha$, we get

$$2\alpha \int_0^1 du\, u\, J_\nu^2(\alpha u) = u^2 J_\nu'(\alpha u) \frac{d}{du} J_\nu(\alpha u) \bigg|_0^1 - u\, J_\nu(\alpha u) \frac{d}{du} \left(\frac{u}{\alpha} \frac{d}{du} J_\nu(\alpha u)\right) \bigg|_0^1 ,$$

where we need to be careful that $dJ_\nu(\alpha u)/d\alpha = u J_\nu'(\alpha) = \frac{u}{\alpha} \frac{d}{du} J_\nu(\alpha u)$. The lower endpoint vanishes. Using the Bessel equation gives

$$2\alpha \int_0^1 du\, u\, J_\nu^2(\alpha u) = \alpha J_\nu'^2(\alpha) + J_\nu(\alpha) \frac{\alpha^2 - \nu^2}{\alpha} J_\nu(\alpha).$$

Substitute $J_\nu'(\alpha) \to \frac{\nu}{\alpha} J_\nu(\alpha) - J_{\nu+1}(\alpha)$ to get

$$\begin{aligned}
\int_0^1 du\, u\, J_\nu^2(\alpha u) &= \frac{1}{2} \left(\frac{\nu}{\alpha} J_\nu(\alpha) - J_{\nu+1}(\alpha)\right)^2 + \frac{\alpha^2 - \nu^2}{2\alpha^2} J_\nu^2(\alpha) \\
&= \frac{1}{2} J_{\nu+1}^2(\alpha) - \frac{\nu}{\alpha} J_\nu(\alpha) J_{\nu+1}(\alpha) + \frac{1}{2} J_\nu^2(\alpha). \qquad (7)
\end{aligned}$$

If we set α to a zero of J_ν, the last two terms vanish and

$$\int_0^1 du\, u\, J_\nu^2(x_{\nu\, n} u) = \frac{1}{2} J_{\nu+1}^2(x_{\nu\, n} u). \tag{8}$$

If we set α to $x'_{\nu\, n}$, a zero of $J'_\nu = \frac{\nu}{x'_{\nu\, n}} J_\nu(x'_{\nu\, n}) - J_{\nu+1}(x'_{\nu\, n})$, so $J_{\nu+1}(x'_{\nu\, n}) = \frac{\nu}{x'_{\nu\, n}} J_\nu(x'_{\nu\, n})$, and (7) becomes

$$\int_0^1 du\, u\, J_\nu^2(x'_{\nu\, n} u) = \frac{1}{2}\left[1 - \left(\frac{\nu}{x'_{\nu\, n}}\right)^2\right] J_\nu^2(x'_{\nu\, n}) \tag{9}$$