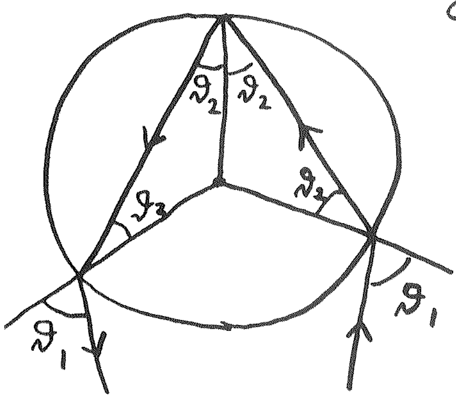
Optics 305. RAINBOW C. Lovelace 9/99



Consider reflection of sunlight by a raindrop. Suppose light hits at ϑ_1 to a radius. The internal angle ϑ_2 is given by Snell

$$n \sin \vartheta_2 = \sin \vartheta_1,$$

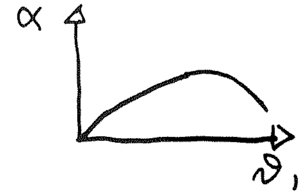
with $n \approx 4/3$ for water. It will again hit the far side of the drop at ϑ_2 to a radius, because the triangle is isosceles. Some of this light will get out, but some will be internally reflected. The rainbow comes from light that undergoes one internal reflection before emerging. The picture shows that the ray is bent by the following angles

$\vartheta_1 - \vartheta_2$ at first refraction (entrance),
$180^\circ - 2\vartheta_2$ at internal reflection,
$\vartheta_1 - \vartheta_2$ at second refraction (exit).

The total deflection is $180^\circ - \alpha$, where $\alpha = 4\vartheta_2 - 2\vartheta_1$.

Now make a little table, using $n = 4/3$.

ϑ_1	ϑ_2	α
0°	0°	0°
30°	22.0°	28.1°
60°	40.5°	42.0°
90°	48.6°	14.4°



We see that, when plotted as a function of ϑ_1, α has a MAXIMUM. 180° is the point opposite the sun, so α is the angle relative to that point. Now there are thousands of raindrops, and many rays hitting each, so the angles ϑ_1 at which sunlight hits them are essentially random. The little graph shows that no light will be reflected beyond α_{max}. Furthermore, many values of ϑ_1 will give angles near α_{max} because the graph is flat there. So we expect a bright circle of angular radius α_{max} around the point opposite the sun, and relative darkness outside it.

Find α_{max}:

$$\vartheta_2 = \sin^{-1}\left(\frac{1}{n}\sin\vartheta_1\right)$$

$$\therefore \frac{d\vartheta_2}{d\vartheta_1} = \frac{\cos\vartheta_1}{\sqrt{n^2 - \sin^2\vartheta_1}}$$

$$\frac{d\alpha}{d\vartheta_1} = 4\frac{d\vartheta_2}{d\vartheta_1} - 2 = 0 \quad \text{at maximum,}$$

so $\dfrac{1-\sin^2\vartheta_1}{n^2-\sin^2\vartheta_1} = \dfrac{1}{4}$, giving

$\sin^2\vartheta_1 = (4-n^2)/3$,

For $n = 4/3$, the maximum is at $\vartheta_1 = 59.4°$, $\vartheta_2 = 40.2°$, and its value is $\alpha_{max} = 4\vartheta_2 - 2\vartheta_1 = 42.0°$.

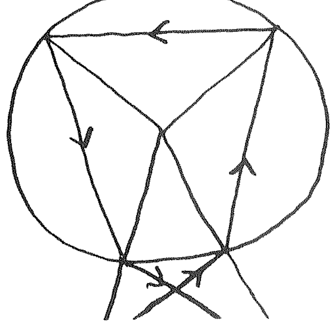
So we predict a bright circle of radius 42° around the point opposite the sun.

However the index of refraction of water depends slightly on the wavelength. More precisely

	λ	n	ϑ_1	ϑ_2	α_{max}
red	700 nm	1.330	59.6°	40.4°	42.5°
violet	400 nm	1.342	58.9°	39.6°	40.8°

So the red circle will have radius 42.5°, but the violet circle only 40.8°.

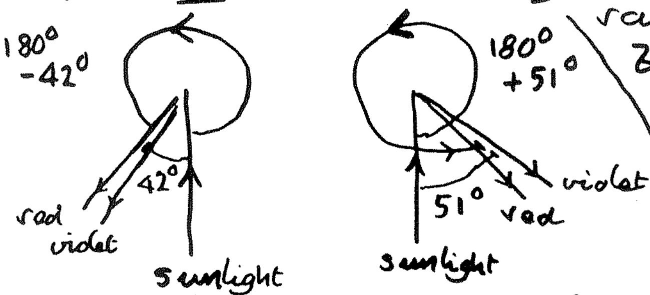
What about the second inverted rainbow, which is sometimes seen above the main one? This comes from light rays which undergo TWO internal reflections before emerging. At each reflection, some light is lost to



refraction, so the second rainbow will be much fainter. All 3 internal triangles in the picture are isosceles, so the second reflection adds another $180° - 2\vartheta_2$, giving a total deflection $180° - \alpha_2$, where $\alpha_2 = 6\vartheta_2 - 2\vartheta_1 - 180°$. The condition for a maximum is now

$$6\frac{\cos\vartheta_1}{\sqrt{n^2-\sin^2\vartheta_1}} - 2 = 0,$$

giving $\sin^2\vartheta_1 = (9-n^2)/8$, and for $n = 4/3$, $\vartheta_1 = 71.8°$, $\vartheta_2 = 45.5°$, $\alpha_2^{max} = 6\vartheta_2 - 2\vartheta_1 - 180° = -51°$. The minus sign means that the light is deflected by more than 180°. The angular radius of the second rainbow will be 51°, but the minus sign will turn it upside down colourwise. This is illustrated in the drawings. Red has smaller n, so is deflected less than violet.



first rainbow showing angle of deflection

second rainbow