ϵ in higher dimension Euclidean and Minkowski spaces

The antisymmetric tensor ϵ_{ijk} defined in three dimensions can be extended to higher dimension D, though only with D indices. To allow for Minkowski as well as Euclidean space, we will distinguish between upper and lower indices, related by the Minkowski metric tensor $g_{\mu\nu}$, which we will take to have only diagonal elements equal to ± 1[1].

We define $\epsilon^{\mu_1\mu_2\cdots\mu_D}$ to be the totally antisymmetric (under interchange of pairs of indices) tensor with $\epsilon^{0\,1\,\cdots\,D-1} = 1$. As the tensor is zero if any two indices are the same, and as there are the same number of indices as there are possible values, the only nonzero values of ϵ are when the indices are a permutation P of the D possible index values, which we take here to run from 0 to $D-1$ (as ordinary Minkowski space will be our usual application). When the indices are a permuation P of these values, the value of ϵ is just the "sign of the permutation P", written $(-1)^P$.

Thus we can write

$$\epsilon^{\mu_1\mu_2\cdots\mu_D} = \sum_{P\in S_D} (-1)^P \prod_{j=0}^{D-1} \delta^{\mu_j}_{Pj}, \tag{1}$$

where S_D is the set of permutations on D objects, here labelled $0,\ldots,D-1$, and Pj is the object P maps object j into. Then we get a basic identity

$$\epsilon^{\mu_1\mu_2\ldots\mu_D}\epsilon^{\nu_1\nu_2\ldots\nu_D} = \sum_{P\in S_D} (-1)^P \prod_{j=0}^{D-1} \delta^{\;\nu_j}_{\mu_{Pj}}. \tag{2}$$

To see that this is true, note that if any two of the μ's or any two of the ν's are equal, both sides are zero because interchanging the two delta functions with the equal values does not change the product but is equivalent to changing the permutation by a transposition, which reverses $(-1)^P$. If the μ's are $P_\mu(0,1,\ldots,D-1)$ and the ν's are $P_\nu(0,1,\ldots,D-1)$, then the left hand side is $(-1)^{P_\mu}(-1)^{P_\nu}$, while the only permutation P which gives a contribution on the right hand side is $P = P_\nu P_\mu^{-1}$, with $(-1)^P = (-1)^{P_\mu}(-1)^{P_\nu}$.

[1]Usually for Minkowski space there will be one time with $g_{00} = 1$ and $D-1$ space dimensions with $g_{ij} = -\delta_{ij}$. For Euclidean space we take all diagonal elements to be $+1$, so g is the unit matrix and there is no distinction between upper and lower indices.

Note that expression 2 doesn't look right, because on the left side we have μ_j as a contravariant (upper) index, while on the left it is a covariant (lower) index. We can make a covariant statement by lowering all the indices, which introduces the product of all the diagonal elements of $g_{\mu\rho}$, which is its determinant $\det(g)$. Thus

$$\epsilon_{\mu_0\mu_1\ldots\mu_{D-1}}\epsilon^{\nu_0\nu_1\ldots\nu_{D-1}} = \det(g)\sum_{P\in S_D}(-1)^P\prod_{j=0}^{D-1}\delta^{\nu_j}_{\mu_{Pj}} \tag{3}$$

Two alternate expressions for the same product of ϵ's are

$$\begin{aligned}
\epsilon_{\mu_0\mu_1\ldots\mu_{D-1}}\epsilon^{\nu_0\nu_1\ldots\nu_{D-1}} &= \det(g)\sum_{P\in S_D}(-1)^P\prod_{j=0}^{D-1}\delta^{\nu_{Pj}}_{\mu_j} && (4)\\
&= \det(g)\sum_{P\in S_D}(-1)^P\prod_{j=0}^{D-1}\delta^{\nu_j}_{P\mu_j}. && (5)
\end{aligned}$$

In the second form we permute the values of the indices ν rather than μ. As a permutation of the delta functions factors in the product makes no difference, for any permutation P'

$$\prod_{j=0}^{D-1}\delta^{\nu_j}_{\mu_{Pj}} = \prod_{j=0}^{D-1}\delta^{\nu_{P'j}}_{\mu_{PP'j}}.$$

For each P in the sum in (3) we can choose $P' = P^{-1}$ to convert it to (4) (renaming the dummy summation $P \to P^{-1}$). In the third form we permute the values of the indices μ rather than *their* indices j. Unless all the μ_j are different the rhs of both (3) and (5) will vanish, and if they are all different, there is a permutation P' with $\mu_j = P'j$ for all j. Then the lower indices in (3) are $P'P_aj$ and those of (5) are $P_bP'j$, where I have renamed the dummy summation variables to distinguish them. In either case only one permutation will give a nonzero contribution, with $P'P_a = P_bP'$, or $P_b = P'P_a(P')^{-1}$. As

$$(-1)^{P_b} = (-1)^{P'P_a(P')^{-1}},$$

we have verified that (5) is equal to (3).

[A flash forward to curved space: In Riemann space the metric tensor $g_{\mu\nu}$ is no longer diagonal nor are its non-zero elements restricted to ± 1. The contravariant tensor is then defined[2] as $\varepsilon^{\mu_1\mu_2\cdots\mu_D} = (|\det(g)|)^{-1/2}\epsilon^{\mu_1\mu_2\cdots\mu_D}$, and the covariant one is $\varepsilon_{\mu_1\mu_2\cdots\mu_D} = \operatorname{sign}(\det(g))(|\det(g)|)^{1/2}\epsilon^{\mu_1\mu_2\cdots\mu_D}$.]

The basic identity (5) is seldom used in its full form, but often used with some indices contracted. Then in the right hand side of (5) only permutations which map the first ($j = 0$) index into itself can contribute. There is also only one value of that index which is unequal to all the others, if they are all different. The remaining permutations are over S_{D-1} acting on the remaining values

$$\epsilon_{\mu\mu_1\cdots\mu_{D-1}}\epsilon^{\mu\nu_1\cdots\nu_{D-1}} = \det(g)\sum_{P\in S_{D-1}}(-1)^P\prod_{j=1}^{D-1}\delta^{\ \nu_j}_{P\mu_j}$$

For example, we derive expressions in four dimensional ordinary Minkowski space. Then

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = -\left(\delta^\nu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma + \delta^\sigma_\alpha\delta^\nu_\beta\delta^\rho_\gamma + \delta^\rho_\alpha\delta^\sigma_\beta\delta^\nu_\gamma - \delta^\rho_\alpha\delta^\nu_\beta\delta^\sigma_\gamma - \delta^\sigma_\alpha\delta^\rho_\beta\delta^\nu_\gamma - \delta^\nu_\alpha\delta^\sigma_\beta\delta^\rho_\gamma\right).$$

If you then contract again, σ with γ, you get

$$\begin{aligned}\epsilon^{\mu\nu\rho\gamma}\epsilon_{\mu\alpha\beta\gamma} &= -\left(\delta^\nu_\alpha\delta^\rho_\beta\delta^\gamma_\gamma + \delta^\gamma_\alpha\delta^\nu_\beta\delta^\rho_\gamma + \delta^\rho_\alpha\delta^\gamma_\beta\delta^\nu_\gamma - \delta^\rho_\alpha\delta^\nu_\beta\delta^\gamma_\gamma - \delta^\gamma_\alpha\delta^\rho_\beta\delta^\nu_\gamma - \delta^\nu_\alpha\delta^\gamma_\beta\delta^\rho_\gamma\right)\\ &= -\left(4\delta^\nu_\alpha\delta^\rho_\beta + \delta^\rho_\alpha\delta^\nu_\beta + \delta^\rho_\alpha\delta^\nu_\beta - 4\delta^\rho_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\rho_\beta - \delta^\nu_\alpha\delta^\rho_\beta\right)\\ &= -2\left(\delta^\nu_\alpha\delta^\rho_\beta - \delta^\rho_\alpha\delta^\nu_\beta\right).\end{aligned}$$

Contracting once or twice again,

$$\begin{aligned}\epsilon^{\mu\nu\rho\gamma}\epsilon_{\mu\nu\beta\gamma} &= -2\left(\delta^\nu_\nu\delta^\rho_\beta - \delta^\rho_\nu\delta^\nu_\beta\right) = -2\left(4\delta^\nu_\nu\delta^\rho_\beta - \delta^\rho_\beta\right) = -6\delta^\rho_\beta,\\ \epsilon^{\mu\nu\rho\gamma}\epsilon_{\mu\nu\rho\gamma} &= -6\delta^\rho_\rho = -24.\end{aligned}$$

1 Uses of ϵ

1.1 Determinants

Consider an $N \times N$ matrix $\mathbf{A}$ with matrix elements $A_{j,k}$. What is its determinant? Two easy answers:

$$\det\mathbf{A} = \epsilon^{\mu_1\mu_2\ldots\mu_N}\prod_j A_{j,\mu_j} = \epsilon^{\mu_1\mu_2\ldots\mu_N}\prod_j A_{\mu_j,j},$$

[2]Except Misner Thorne and Wheeler put the sign(det(g)) on the contravariant ε rather than the covariant one, as do Bjorken and Drell even in flat Minkowski space.

where we are using an N dimensional Euclidean ϵ. Of course summation over each μ_j is understood. That the two expressions are equivalent is shown by the same argument that showed the equivalence of (3) and (4), based on the product being unchanged by permuting the factors. Many properties of the determinant follow directly (a) the expansion by minors[3] (b) antisymmetry under interchange of two rows or two columns, (c) linearity in each row and in each column. Furthermore, we may note that

$$\epsilon^{\mu_1\mu_2\ldots\mu_N}\prod_j A_{\mu_j,\nu_j} = \det\mathbf{A}\,\epsilon^{\nu_1\nu_2\ldots\nu_N}, \tag{6}$$

because the expression is totally antisymmetric under permutations of the ν's and is $\det\mathbf{A}$ when they are in order. From this expression, it is obvious that the determinant of a product is the product of the determinants:

$$\begin{aligned}\det\mathbf{AB} &= \epsilon^{\mu_1\mu_2\ldots\mu_N}\prod_j(AB)_{\mu_j,j} = \epsilon^{\mu_1\mu_2\ldots\mu_N}\prod_j A_{\mu_j,\nu_j}B_{\nu_j,j}\\ &= \det\mathbf{A}\,\epsilon^{\nu_1\nu_2\ldots\nu_N}\prod_j B_{\nu_j,j} = \det\mathbf{A}\det\mathbf{B}.\end{aligned}$$

[3] Alias cofactor expansion, expansion by row or column.