ϵ_{ijk} and cross products in 3-D Euclidean space

These are some notes on the use of the antisymmetric symbol ϵ_{ijk} for expressing cross products. This is an extremely powerful tool for manipulating cross products and their generalizations in higher dimensions, and although many low level courses avoid the use of ϵ, I think this is a mistake and I want you to become proficient with it.

In a cartesian coordinate system a vector $\vec{V}$ has components V_i along each of the three orthonormal basis vectors $\hat{e}_i$, or $\vec{V} = \sum_i V_i \hat{e}_i$. The dot product of two vectors, $\vec{A} \cdot \vec{B}$, is bilinear and can therefore be written as

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= (\sum_i A_i \hat{e}_i) \cdot \sum_j B_j \hat{e}_j && (1)\\
&= \sum_i \sum_j A_i B_j \hat{e}_i \cdot \hat{e}_j && (2)\\
&= \sum_i \sum_j A_i B_j \delta_{ij}, && (3)
\end{aligned}$$

where the Kronecker delta δ_{ij} is defined to be 1 if $i = j$ and 0 otherwise. As the basis vectors $\hat{e}_k$ are orthonormal, *i.e.* orthogonal to each other and of unit length, we have $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$.

Doing a sum over an index j of an expression involving a δ_{ij} is very simple, because the only term in the sum which contributes is the one with $j = i$. Thus $\sum_j F(i,j)\delta_{ij} = F(i,i)$, which is to say, one just replaces j with i in all the other factors, and drops the δ_{ij} and the summation over j. So we have $\vec{A} \cdot \vec{B} = \sum_i A_i B_i$, the standard expression for the dot product[1]

We now consider the cross product of two vectors, $\vec{A} \times \vec{B}$, which is also a bilinear expression, so we must have $\vec{A} \times \vec{B} = (\sum_i A_i \hat{e}_i) \times (\sum_j B_j \hat{e}_j) = \sum_i \sum_j A_i B_j (\hat{e}_i \times \hat{e}_j)$. The cross product $\hat{e}_i \times \hat{e}_j$ is a vector, which can therefore be written as $\vec{V} = \sum_k V_k \hat{e}_k$. But the vector result depends also on the two input vectors, so the coefficients V_k really depend on i and j as well. Define them to be ϵ_{ijk}, so

$$\hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{kij} \hat{e}_k.$$

[1] Note that this only holds because we have expressed our vectors in terms of *orthonormal* basis vectors.

It is easy to evaluate the 27 coefficients ϵ_{kij}, because the cross product of two orthogonal unit vectors is a unit vector orthogonal to both of them. Thus $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, so $\epsilon_{312} = 1$ and $\epsilon_{k12} = 0$ if $k = 1$ or 2. Applying the same argument to $\hat{e}_2 \times \hat{e}_3$ and $\hat{e}_3 \times \hat{e}_1$, and using the antisymmetry of the cross product, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, we see that

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1; \qquad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1,$$

and $\epsilon_{ijk} = 0$ for all other values of the indices, *i.e.* $\epsilon_{ijk} = 0$ whenever any two of the indices are equal. Note that ϵ changes sign not only when the last two indices are interchanged (a consequence of the antisymmetry of the cross product), but whenever *any* two of its indices are interchanged. Thus ϵ_{ijk} is zero unless $(1, 2, 3) \to (i, j, k)$ is a permutation, and is equal to the sign of the permutation if it exists.

Now that we have an expression for $\hat{e}_i \times \hat{e}_j$, we can evaluate

$$\vec{A} \times \vec{B} = \sum_i \sum_j A_i B_j (\hat{e}_i \times \hat{e}_j) = \sum_i \sum_j \sum_k \epsilon_{kij} A_i B_j \hat{e}_k. \tag{4}$$

Much of the usefulness of expressing cross products in terms of ϵ's comes from the identity

$$\sum_k \epsilon_{kij} \epsilon_{k\ell m} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}, \tag{5}$$

which can be shown as follows. To get a contribution to the sum, k must be different from the unequal indices i and j, and also different from ℓ and m. Thus we get 0 unless the pair (i, j) and the pair (ℓ, m) are the same pair of different indices. There are only two ways that can happen, as given by the two terms, and we only need to verify the coefficients. If $i = \ell$ and $j = m$, the two ϵ's are equal and the square is 1, so the first term has the proper coefficient of 1. The second term differs by one transposition of two indices on one epsilon, so it must have the opposite sign.

We now turn to some applications. Let us first evaluate

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \sum_i A_i \sum_{jk} \epsilon_{ijk} B_j C_k = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k. \tag{6}$$

Note that $\vec{A} \cdot (\vec{B} \times \vec{C})$ is, up to sign, the volume of the parallelopiped formed by the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. From the fact that the ϵ changes sign under transpositions of any two indices, we see that the same is true for transposing the vectors, so that

$$\begin{aligned}\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) &= \vec{B}\cdot(\vec{C}\times\vec{A}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) \\ &= \vec{C}\cdot(\vec{A}\times\vec{B}) = -\vec{C}\cdot(\vec{B}\times\vec{A}).\end{aligned}$$

Now consider $\vec{V} = \vec{A}\times(\vec{B}\times\vec{C})$. Using our formulas,

$$\vec{V} = \sum_{ijk}\epsilon_{kij}\hat{e}_k A_i(\vec{B}\times\vec{C})_j = \sum_{ijk}\epsilon_{kij}\hat{e}_k A_i \sum_{lm}\epsilon_{jlm}B_l C_m.$$

Notice that the sum on j involves only the two epsilons, and we can use

$$\sum_j \epsilon_{kij}\epsilon_{jlm} = \sum_j \epsilon_{jki}\epsilon_{jlm} = \delta_{kl}\delta_{im} - \delta_{km}\delta_{il}.$$

Thus

$$\begin{aligned}V_k &= \sum_{ilm}(\sum_j \epsilon_{kij}\epsilon_{jlm})A_i B_l C_m = \sum_{ilm}(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})A_i B_l C_m \\ &= \sum_{ilm}\delta_{kl}\delta_{im}A_i B_l C_m - \sum_{ilm}\delta_{km}\delta_{il}A_i B_l C_m \\ &= \sum_i A_i B_k C_i - \sum_i A_i B_i C_k = \vec{A}\cdot\vec{C}\, B_k - \vec{A}\cdot\vec{B}\, C_k,\end{aligned}$$

so

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,\vec{A}\cdot\vec{C} - \vec{C}\,\vec{A}\cdot\vec{B}. \tag{7}$$

This is sometimes known as the **bac-cab** formula.

Exercise: Using (5) for the manipulation of cross products, show that

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \vec{A}\cdot\vec{C}\vec{B}\cdot\vec{D} - \vec{A}\cdot\vec{D}\vec{B}\cdot\vec{C}.$$

The determinant of a matrix can be defined using the ϵ symbol. For a 3×3 matrix A,

$$\det A = \sum_{ijk}\epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \sum_{ijk}\epsilon_{ijk}A_{i1}A_{j2}A_{k3}.$$

From the second definition, we see that the determinant is the volume of the parallelopiped formed from the images under the linear map A of the three unit vectors $\hat{e}_i$, as

$$(A\hat{e}_1)\cdot((A\hat{e}_2)\times(A\hat{e}_3)) = \det A.$$

In higher dimensions, the cross product is not a vector, but there is a generalization of ϵ which remains very useful. In an n-dimensional space, $\epsilon_{i_1 i_2 \dots i_n}$ has n indices and is defined as the sign of the permutation $(1, 2, \dots, n) \to (i_1 i_2 \dots i_n)$, if the indices are all unequal, and zero otherwise. The analog of (5) has $(n-1)!$ terms from all the permutations of the unsummed indices on the second ϵ. The determinant of an $n \times n$ matrix is defined as

$$\det A = \sum_{i_1, \dots, i_n} \epsilon_{i_1 i_2 \dots i_n} \prod_{p=1}^{n} A_{p, i_p}.$$