Physics 507 Homework #5
Due: Thursday, Oct. 7, 2010

5.1 Consider a particle constrained to move on the surface described in cylindrical coordinates by $z = \alpha r^3$, subject to a constant gravitational force $\vec{F} = -mg\hat{e}_z$. Find the Lagrangian, two conserved quantities, and reduce the problem to a one dimensional problem. What is the condition for circular motion at constant r?

5.2 Suppose a particle of mass m moves under the influence of a power-law central force, $\vec{F} = -cr^p\hat{e}_r$, and is observed to have an orbit which is a circle of radius R *passing through* the point of attraction.

Find what values the power p could be, what is the angular momentum about the *center of force*, and what is the energy relative to $U(\infty)$.

How do $\dot{\theta}$, $\dot{y}$, and $\dot{x}$ behave as the particle approaches the origin, as a function of r as $r \to 0$?. Is this consistent with x taking its minimum value at that point?

Hamilton's Principle tells us that the motion of a particle is determined by the action functional being stationary under small variations of the path Γ in extended configuration space $(t, \vec{x})$. The unsymmetrical treatment of t and $\vec{x}(t)$ is not suitable for relativity, but we may still associate an action with each path, which we can parameterize with λ, so Γ is the trajectory $\lambda \to (t(\lambda), \vec{x}(\lambda))$.

In the general relativistic treatment of a particle's motion in a gravitational field, the action is given by $mc^2\Delta\tau$, where $\Delta\tau$ is the elapsed proper time, $\Delta\tau = \int d\tau$. But distances and time intervals are measured with a spatial varying metric $g_{\mu\nu}$, with μ and ν ranging from 0 to 3, with the zeroth component referring to time. The four components of extended configuration space are written x^μ, with a superscript rather than a subscript, and $x^0 = ct$. The other three x^μ can be generalized coordinates, as long as $g_{\mu\nu}$ is appropriate. In the next problem they are similar to spherical coordinates. The gravitational field is described by the space-time dependence of the metric $g_{\mu\nu}(x^\rho)$. In this language, an infinitesimal element of the path of a particle corresponds to a proper time $d\tau = (1/c)\sqrt{\sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu}$, so

$$S = mc^2\Delta\tau = mc\int d\lambda \sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}.$$

This is in preparation for the next problem *which is worth 20 points, twice normal.*

5.3 In problem 2.12 we learned that the general-relativistic motion of a particle in a gravitational field is given by Hamilton's variational principle on the path $x^\mu(\lambda)$ with the action

$$S = \int d\lambda \mathcal{L} \quad \text{with} \quad \mathcal{L} = mc\sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}},$$

where we may freely choose the path parameter λ to be the proper time (after doing the variation), so that the $\sqrt{}$ is c, the speed of light.

The gravitational field of a static point mass M is given by the Schwartzschild metric

$$g_{00} = 1 - \frac{2GM}{rc^2}, \quad g_{rr} = -1 \Big/ \left(1 - \frac{2GM}{rc^2}\right), \quad g_{\theta\theta} = -r^2, \quad g_{\phi\phi} = -r^2\sin^2\theta,$$

where all other components of $g_{\mu\nu}$ are zero. Treating the four $x^\mu(\lambda)$ as the coordinates, with λ playing the role of time, find the four conjugate momenta p_μ, show that p_0 and $p_\phi = L$ are constants, and use the freedom to choose

$$\lambda = \tau = \frac{1}{c}\int\sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

to show $m^2c^2 = \sum_{\mu\nu} g^{\mu\nu}p_\mu p_\nu$, where $g^{\mu\nu}$ is the inverse matrix to $g_{\alpha\beta}$. Use this to show that

$$\frac{dr}{d\tau} = \sqrt{\kappa - \left(-\frac{2GM}{r} + \frac{L^2}{m^2r^2} - \frac{2GML^2}{m^2r^3c^2}\right)},$$

where κ is a constant. For an almost circular orbit at the minimum $r = a$ of the effective potential this implies, show that the precession of the perihelion is $6\pi GM/ac^2$.

Find the rate of precession for Mercury, with $G = 6.67 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$, $M = 1.99 \times 10^{30}\,\text{kg}$ and $a = 5.79 \times 10^{10}\,\text{m}$, per revolution, and also per century, using the period of the orbit as 0.241 years.