Notes to be added at the end of Section 5.2

In this section we will explore the effects of driving forces on oscillators. For simplicity let us consider a damped oscillator with one degree of freedom, with a driving force $F(t)$:

$$m\ddot{x}(t) + R\dot{x}(t) + kx(t) = F(t).$$

For this linear oscillator, we can solve by Fourier transform. Writing $x(t) = \int_{-\infty}^{\infty} \tilde{x}(\omega)e^{-i\omega t}\, d\omega$, we find $(-m\omega^2 - iR\omega + k)\,\tilde{x}(\omega) = \tilde{F}(\omega)$, where the Fourier transformed force is $\tilde{F}(\omega) := \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{i\omega t}dt$.

Without any forces, we have solutions for $\omega^2 + 2i\rho\omega - \omega_0^2 = 0$, (where $\omega_0 := \sqrt{k/m}$, $\rho = R/2m$), so the solutions are at $\omega = -i\rho \pm \bar{\omega}$, with $\bar{\omega} = \sqrt{\omega_0^2 - \rho^2}$. Due to the negative imaginary part of either of these ωs, the unforced oscillations will decay with time. If we do have a forcing function, however, we have an inhomogeneous solution (with $\tilde{f} = \tilde{F}/m$)

$$\tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 - 2i\rho\omega}.$$

As $x(t)$ and $f(t)$ are real-valued functions of time, the fourier transforms must satisfy $\tilde{x}^*(\omega) = \tilde{x}(-\omega)$, $\tilde{f}^*(\omega) = \tilde{f}(-\omega)$, and

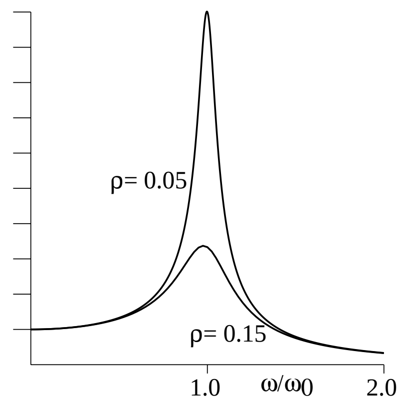
$$\begin{aligned} x(t) &= \int_0^{\infty} d\omega \left(\tilde{x}(\omega)e^{-i\omega t} + \tilde{x}^*(\omega)e^{i\omega t}\right) = 2\Re \int_0^{\infty} d\omega \tilde{x}(\omega)e^{-i\omega t} \\ &= 2\Re \int_0^{\infty} d\omega \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 - 2i\rho\omega} e^{-i\omega t}. \end{aligned}$$

If we consider a forcing function of only one positive frequency, say $\tilde{f}(\omega) = a\delta(\omega - \omega_{\text{ex}})$ for $\omega \geq 0$, we have

$$x(t) = 2\Re \frac{a}{\omega_0^2 - \omega_{\text{ex}}^2 - 2i\rho\omega_{\text{ex}}} e^{-i\omega_{\text{ex}} t},$$

with amplitude

$$
\begin{aligned}
A &= \left|\frac{2a}{\omega_0^2-\omega_{\rm ex}^2-2i\rho\omega_{\rm ex}}\right| \\
&= \frac{2|a|}{\sqrt{\left(\omega_0^2-\omega_{\rm ex}^2\right)^2+4\rho^2\omega_{\rm ex}^2}}.
\end{aligned}
$$



We see that the response in the frequency domain is proportional to the force, with a frequency dependence which is sharply peaked if the damping coefficient is small compared to the natural frequency, $\rho \ll \omega_0$.

If we ask in the temporal domain, what is the effect on $x(t)$ of a force $f(t')$, we have

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} d\omega\,\frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\,\frac{f(t')}{\omega_0^2-\omega^2-2i\rho\omega}e^{i\omega(t-t')} \\
&= \int_{-\infty}^{\infty} dt'\,G(t-t')\frac{f(t')}{m}, \qquad (1)
\end{aligned}
$$

where

$$
G(t-t') := \frac{1}{2\pi}\int d\omega\,\frac{e^{-i\omega(t-t')}}{\omega_0^2-\omega^2-2i\rho\omega} \qquad (2)
$$

is the temporal Green's function. Note that it would appear from (1) that effects could preceed causes, as the integral is over all t', including times after t, but in fact G vanishes there. For $t' > t$ we may evaluate (2) by closing the integration contour in the upper half plane, for the exponential will vanish for large $\Im\omega > 0$ when $t < t'$. As the integrand is analytic in the upper half plane, the contour integral vanishes, and $G(t-t') = 0$ for $t' > t$. On the other hand, for $\Delta t = t - t' > 0$, the integration contour may be closed in the lower half plane, picking up the residues from the poles at $\omega = -i\rho \pm \bar\omega$. The residue there is $\mp e^{-\rho\,\Delta t}e^{\mp i\bar\omega\Delta t}/\bar\omega$, so

$$
G(\Delta t) = \frac{1}{\bar\omega}e^{-\rho\,\Delta t}\sin\bar\omega\Delta t.
$$

0.0.1 Weakly nonlinear oscillating systems

The oscilator we just considered could be solved exactly because it is a linear system. The equation of motion is a linear operator (including time derivative operators) acting on the dynamical variable $x(t)$, set equal to a forcing term which is a given function of time. Most systems, however, are not exactly linear. If the equation of motion is close to linear, we might imagine a perturbative calculation in which we bring the difference from linearity, considered small, to the right hand side, evaluate it in the linear approximation, and consider it a forcing term. For example, we are quite used to the idea that a pendulum may be approximated by a harmonic oscillator. A forced, linearly damped pendulum has an equation of motion

$$m\ell^2\ddot{\theta} + R\dot{\theta} + mg\ell\sin\theta = F(t),$$

which in the approximation $\sin\theta \approx \theta$ reduces to the harmonic oscillator we just considered. More precisely, we can write

$$\ddot{\theta} + 2\rho\dot{\theta} + \omega_0^2\theta = f(t) - \omega_0^2(\sin\theta - \theta),$$

where $\rho = R/2m\ell^2$, $\omega_0 = \sqrt{g/\ell}$ and $f(t) = F(t)/m\ell^2$. If the forcing function $f(t)$ and the oscillations are small ($\theta \ll \pi$), we can imagine a sequence of approximations, first evaluating $\theta(t)$ dropping the $(\sin\theta - \theta)$ term, and then evaluating the $n+1$'st approximation to $\theta(t)$ by using the n'th approximation to evaluate $(\sin\theta - \theta)(t)$ as a forcing term.

We will return to this issue, discussing both how this works and why it may not be the ideal way to do a perturbative expansion, in Chapter 7.