Chapter 5

Small Oscillations

5.1 Small oscillations about stable equilibrium

Consider a situation with N unconstrained generalized coordinates q_i described by a mass matrix $M_{ij}(\{q_k\})$ and a potential $U(\{q_i\})$, and suppose that U has a local minimum at some point in configuration space, $q_i = q_{i0}$. Then this point is a stable equilibrium point, for the generalized force at that point is zero, and if the system is placed nearly at rest near that point, it will not have enough energy to move far away from that point. We may study the behavior of such motions by expanding the potential[1] in Taylor's series expansion in the deviations $\eta_i = q_i - q_{i0}$,

$$U(q_1, \ldots, q_N) = U(q_{i0}) + \sum_i \left.\frac{\partial U}{\partial q_i}\right|_0 \eta_i + \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_0 \eta_i \eta_j + \ldots .$$

The constant $U(q_{i0})$ is of no interest, as only changes in potential matter, so we may as well set it to zero. In the second term, $-\left.\partial U/\partial q_i\right|_0$ is the generalized force at the equilibrium point, so it is zero. Thus the leading term in the expansion is the quadratic one, and we may approximate

$$U(\{q_i\}) = \frac{1}{2}\sum_{ij} A_{ij}\eta_i\eta_j, \quad \text{with} \quad A_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_0 . \tag{5.1}$$

Note that A is a constant symmetric real matrix.

[1] assumed to have continuous second derivatives.

The kinetic energy $T = \frac{1}{2}\sum M_{ij}\dot{\eta}_i\dot{\eta}_j$ is already second order in the small variations from equilibrium, so we may evaluate M_{ij}, which in general can depend on the coordinates q_i, at the equilibrium point, ignoring any higher order changes. Thus M_{ij} is a constant. Thus both the kinetic and potential energies are quadratic forms in the displacement η, which we think of as a vector in N-dimensional space. Thus we can write the energies in matrix form

$$T = \frac{1}{2}\dot{\eta}^T \cdot M \cdot \dot{\eta}, \qquad U = \frac{1}{2}\eta^T \cdot A \cdot \eta. \tag{5.2}$$

A and M are real symmetric matrices, and because any displacement corresponds to positive kinetic and nonnegative potential energies, they are positive (semi)definite matrices, meaning that all their eigenvalues are greater than zero, except that A may also have eigenvalues equal to zero (these are directions in which the stability is neutral to lowest order, but may be determined by higher order terms in the displacement).

Lagrange's equation of motion

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_i} - \frac{\partial L}{\partial \eta_i} = \frac{d}{dt}M \cdot \dot{\eta} + A \cdot \eta = M \cdot \ddot{\eta} + A \cdot \eta \tag{5.3}$$

is not necessarily diagonal in the coordinate η. We shall use the fact that any real symmetric matrix can be diagonalized by a similarity transformation with an orthogonal matrix to reduce the problem to a set of independant harmonic oscillators. While both M and A can be diagonalized by an orthogonal transformation, they can not necessarily be diagonalized by the *same* one, so our procedure will be in steps:

1. Diagonalize M with an orthogonal transformation $\mathcal{O}_1$, transforming the coordinates to a new set $x = \mathcal{O}_1 \cdot \eta$.

2. Scale the x coordinates to reduce the mass matrix to the identity matrix. The new coordinates will be called y.

3. Diagonalize the new potential energy matrix with another orthogonal matrix $\mathcal{O}_2$, giving the final set of coordinates, $\xi = \mathcal{O}_2 \cdot y$. Note this transformation leaves the kinetic energy matrix diagonal because the identity matrix is unaffected by similarity transformations.

The ξ are **normal modes**, modes of oscillation which are independent in the sense that they do not affect each other.

Let us do this in more detail. We are starting with the coordinates η and the real symmetric matrices A and M, and we want to solve the equations $M \cdot \ddot{\eta} + A \cdot \eta = 0$. In our first step, we use the matrix $\mathcal{O}_1$, which linear algebra guarantees exists, that makes $m = \mathcal{O}_1 \cdot M \cdot \mathcal{O}_1^{-1}$ diagonal. Note $\mathcal{O}_1$ is time-independent, so defining $x_i = \sum_j \mathcal{O}_{1\,ij}\eta_j$ also gives $\dot{x}_i = \sum_j \mathcal{O}_{1\,ij}\dot{\eta}_j$, and

$$\begin{aligned} T &= \frac{1}{2}\dot{\eta}^T \cdot M \cdot \dot{\eta} \\ &= \frac{1}{2}\dot{\eta}^T \cdot \left(\mathcal{O}_1^{-1} \cdot m \cdot \mathcal{O}_1\right) \cdot \dot{\eta} \\ &= \frac{1}{2}\left(\dot{\eta}^T \cdot \mathcal{O}_1^T\right) \cdot m \cdot (\mathcal{O}_1 \cdot \dot{\eta}) \\ &= \frac{1}{2}\left(\mathcal{O}_1 \cdot \dot{\eta}\right)^T \cdot m \cdot (\mathcal{O}_1 \cdot \dot{\eta}) \\ &= \frac{1}{2}\dot{x}^T \cdot m \cdot \dot{x}. \end{aligned}$$

Similarly the potential energy becomes $U = \frac{1}{2}x^T \cdot \mathcal{O}_1 \cdot A \cdot \mathcal{O}_1^{-1} \cdot x$. We know that the matrix m is diagonal, and the diagonal elements m_{ii} are all strictly positive. To begin the second step, define the diagonal matrix $S_{ij} = \sqrt{m_{ii}}\delta_{ij}$ and new coordinates $y_i = S_{ii}x_i = \sum_j S_{ij}x_j$, or $y = S \cdot x$. Now $m = S^2 = S^T \cdot S$, so $T = \frac{1}{2}\dot{x}^T \cdot m \cdot \dot{x} = \frac{1}{2}\dot{x}^T \cdot S^T \cdot S \cdot \dot{x} = \frac{1}{2}\left(S \cdot \dot{x}\right)^T \cdot S \cdot \dot{x} = \frac{1}{2}\dot{y}^T \cdot \dot{y}$. In terms of y, the potential energy is $U = \frac{1}{2}y^T \cdot B \cdot y$, where

$$B = S^{-1} \cdot \mathcal{O}_1 \cdot A \cdot \mathcal{O}_1^{-1} \cdot S^{-1}$$

is still a symmetric matrix[2].

Finally, let $\mathcal{O}_2$ be an orthogonal matrix which diagonalizes B, so $C = \mathcal{O}_2 \cdot B \cdot \mathcal{O}_2^{-1}$ is diagonal, and let $\xi = \mathcal{O}_2 \cdot y$. Just as in the first step,

$$U = \frac{1}{2}\xi^T \cdot \mathcal{O}_2 \cdot B \cdot \mathcal{O}_2^{-1} \cdot \xi = \frac{1}{2}\xi^T \cdot C \cdot \xi,$$

while the kinetic energy

$$T = \frac{1}{2}\dot{y}^T \cdot \dot{y} = \frac{1}{2}\dot{y}^T \cdot \mathcal{O}_2^T \cdot \mathcal{O}_2 \cdot \dot{y} = \frac{1}{2}\dot{\xi}^T \cdot \dot{\xi}$$

is still diagonal. Because the potential energy must still be nonnegative, all the diagonal elements C_{ii} are nonnegative, and we will call them $\omega_i := \sqrt{C_{ii}}$.

[2] $\mathcal{O}_1 \cdot A \cdot \mathcal{O}_1^{-1}$ is symmetric because A is and $\mathcal{O}_1$ is orthogonal, so $\mathcal{O}_1^{-1} = \mathcal{O}_1^T$.

Then

$$T = \frac{1}{2}\sum_j \dot{\xi}_j^2, \qquad U = \frac{1}{2}\sum_j \omega_j^2 \xi_j^2, \qquad \ddot{\xi}_j + \omega_j^2 \xi_j = 0,$$

so we have N independent harmonic oscillators with the solutions

$$\xi_j = \text{ Re } a_j e^{i\omega_j t},$$

with some arbitrary complex numbers a_j.

To find what the solution looks like in terms of the original coordinates q_i, we need to undo all these transformations. As $\xi = \mathcal{O}_2 \cdot y = \mathcal{O}_2 \cdot S \cdot x = \mathcal{O}_2 \cdot S \cdot \mathcal{O}_1 \cdot \eta$, we have

$$q = q_0 + \mathcal{O}_1^{-1} \cdot S^{-1} \cdot \mathcal{O}_2^{-1} \cdot \xi.$$

We have completely solved this very general problem in small oscillations, at least in the sense that we have reduced it to a solvable problem of diagonalizing symmetric real matrices. What we have done may appear abstract and formal and devoid of physical insight, but it is a general algorithm which *will work* on a very wide class of problems of small oscillations about equilibrium. In fact, because diagonalizing matrices is something for which computer programs are available, this is even a practical method for solving such systems, even if there are dozens of interacting particles.

5.1.1 Molecular Vibrations

Consider a molecule made up of n atoms. We need to choose the right level of description to understand low energy excitations. We do not want to describe the molecule in terms of quarks, gluons, and leptons. Nor do we need to consider all the electronic motion, which is governed by quantum mechanics. The description we will use, called the **Born-Oppenheimer** approximation, is to model the nuclei as classical particles. The electrons, which are much lighter, move around much more quickly and cannot be treated classically; we assume that for any given configuration of the nuclei, the electrons will almost instantaneously find a quantum-mechanical ground state, which will have an energy which depends on the current positions of the nuclei. This is then a potential energy when considering the nuclear motion. The nuclei themselves will be considered point particles, and we

will ignore internal quantum-mechanical degrees of freedom such as nuclear spins. So we are considering n point particles moving in three dimensions, with some potential about which we know only qualitative features. There are $3n$ degrees of freedom. Of these, 3 are the center of mass motion, which, as there are no external forces, is simply motion at constant velocity. Some of the degrees of freedom describe rotational modes, *i.e.* motions that the molecule could have as a rigid body. For a generic molecule this would be three degrees of freedom, but if the equilibrium configuration of the molecule is linear, rotation about that line is not a degree of freedom, and so only two of the degrees of freedom are rotations in that case. The remaining degrees of freedom, $3n-6$ for noncollinear and $3n-5$ for collinear molecules, are **vibrations**.

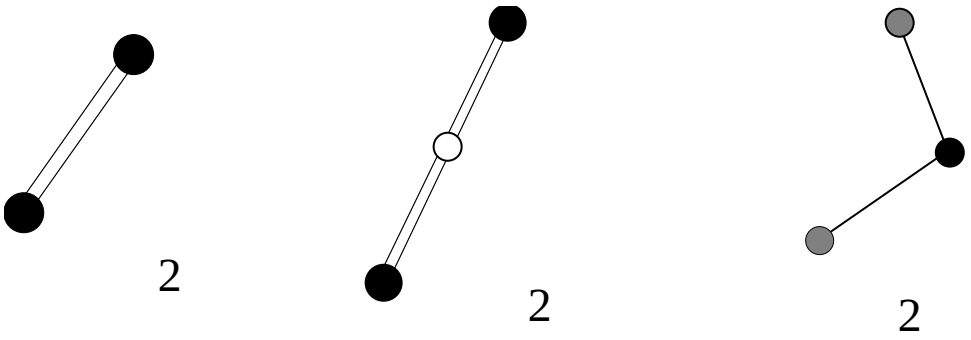


Figure 5.1: Some simple molecules in their equilibrium positions.

For a collinear molecule, it makes sense to divide the vibrations into transverse and longitudinal ones. Considering motion in one dimension only, the nuclei have n degrees of freedom, one of which is a center-of-mass motion, leaving $n-1$ longitudinal vibrations. So the remaining $(3n-5)-(n-1) = 2(n-2)$ vibrational degrees of freedom are transverse vibrational modes. There are no such modes for a diatomic molecule.

Example: CO_2

Consider first the CO_2 molecule. As it is a molecule, there must be a position of stable equilibrium, and empirically we know it to be collinear and symmetric, which one might have guessed. We will first consider only collinear motions of the molecule. If the oxygens have coordinates q_1 and q_2, and the carbon q_3, the potential depends on $q_1 - q_3$ and $q_2 - q_3$ in the same way, so

the equilibrium positions have $q_2 - q_3 = -(q_1 - q_3) = b$. Assuming no direct force between the two oxygen molecules, the one dimensional motion may be described near equilibrium by

$$\begin{aligned} U &= \frac{1}{2}k(q_3 - q_1 - b)^2 + \frac{1}{2}k(q_2 - q_3 - b)^2 \\ T &= \frac{1}{2}m_O\dot{q}_1^2 + \frac{1}{2}m_O\dot{q}_2^2 + \frac{1}{2}m_C\dot{q}_3^2. \end{aligned}$$

We gave our formal solution in terms of displacements from *the* equilibrium position, but we now have a situation in which there is no single equilibrium position, as the problem is translationally invariant, and while equilibrium has constraints on the differences of q's, there is no constraint on the center of mass. We can treat this in two different ways:

1. Explicitly fix the center of mass, eliminating one of the degrees of freedom.

2. Pick arbitrarily *an* equilibrium position. While the deviations of the center-of-mass position from the equilibrium is not confined to small excursions, the quadratic approximation is still exact.

First we follow the first method. We can always work in a frame where the center of mass is at rest, at the origin. Then $m_O(q_1 + q_2) + m_C q_3 = 0$ is a constraint, which we must eliminate. We can do so by dropping q_3 as an independant degree of freedom, and we have, in terms of the two displacements from equilibrium $\eta_1 = q_1 + b$ and $\eta_2 = q_2 - b$, $q_3 = -(\eta_1 + \eta_2)m_O/m_C$, and

$$\begin{aligned} T = \frac{1}{2}m_O(\dot{\eta}_1^2 + \dot{\eta}_2^2) + \frac{1}{2}m_C\dot{\eta}_3^2 &= \frac{1}{2}m_O\left[\dot{\eta}_1^2 + \dot{\eta}_2^2 + \frac{m_O}{m_C}(\dot{\eta}_1 + \dot{\eta}_2)^2\right] \\ &= \frac{1}{2}\frac{m_O^2}{m_C}\begin{pmatrix}\dot{\eta}_1 & \dot{\eta}_2\end{pmatrix}\begin{pmatrix}1 + m_C/m_O & 1 \\ 1 & 1 + m_C/m_O\end{pmatrix}\begin{pmatrix}\dot{\eta}_1 \\ \dot{\eta}_2\end{pmatrix}. \end{aligned}$$

Now T is not diagonal, or more precisely M isn't. We must find the orthogonal matrix $\mathcal{O}_1$ such that $\mathcal{O}_1 \cdot M \cdot \mathcal{O}_1^{-1}$ is diagonal. We may assume it to be a rotation, which can only be

$$\mathcal{O} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

for some value of θ. It is worthwhile to derive a formula for diagonalizing a general real symmetric 2×2 matrix and then plug in our particular form. Let

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad \text{and} \quad \mathcal{O} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix},$$

where we have abbreviated $s = \sin\theta, c = \cos\theta$. We will require the matrix element $m_{12} = (\mathcal{O} \cdot M \cdot \mathcal{O}^{-1})_{12} = 0$, because m is diagonal. This determines θ:

$$\begin{aligned} \mathcal{O} \cdot M \cdot \mathcal{O}^{-1} &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\ &= \begin{pmatrix} c & -s \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & as + bc \\ \cdot & bs + cd \end{pmatrix} = \begin{pmatrix} \cdot & acs + bc^2 - bs^2 - scd \\ \cdot & \cdot \end{pmatrix} \end{aligned}$$

where we have placed a $\cdot$ in place of matrix elements we don't need to calculate. Thus the condition on θ is

$$(a-d)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) = 0 = \frac{1}{2}(a-d)\sin 2\theta + b\cos 2\theta,$$

or

$$\tan 2\theta = \frac{-2b}{a-d}.$$

Notice this determines 2θ only modulo π, and therefore θ modulo 90°, which ought to be expected, as a rotation through 90° only interchanges axes and reverses directions, both of which leave a diagonal matrix diagonal.

In our case $a = d$, so $\tan 2\theta = \infty$, and $\theta = \pi/4$. As $x = \mathcal{O}_1\eta$,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_1 - \eta_2 \\ \eta_1 + \eta_2 \end{pmatrix},$$

and inversely

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{pmatrix}.$$

Then

$$T = \frac{1}{2} m_O \left[\frac{(\dot{x}_1 + \dot{x}_2)^2}{2} + \frac{(\dot{x}_1 - \dot{x}_2)^2}{2} + \frac{m_O}{m_C} (\sqrt{2}\dot{x}_2)^2 \right]$$

$$
\begin{aligned}
&= \frac{1}{2}m_O\dot{x}_1^2 + \frac{1}{2}m_O\left(1+\frac{2m_O}{m_C}\right)\dot{x}_2^2 \\
U &= \frac{1}{2}k(q_3-q_1-b)^2 + \frac{1}{2}k(q_2-q_3-b)^2 \\
&= \frac{1}{2}k\left[\left(\eta_1+\frac{m_O}{m_C}(\eta_1+\eta_2)\right)^2 + \left(\eta_2+\frac{m_O}{m_C}(\eta_1+\eta_2)\right)^2\right] \\
&= \frac{1}{2}k\left[\eta_1^2+\eta_2^2+\frac{2m_O^2}{m_C^2}(\eta_1+\eta_2)^2 + \frac{2m_O}{m_C}(\eta_1+\eta_2)^2\right] \\
&= \frac{1}{2}k\left[x_1^2+x_2^2+\frac{4m_O}{m_C^2}(m_O+m_C)x_2^2\right] \\
&= \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(\frac{m_C+2m_O}{m_C}\right)^2 x_2^2.
\end{aligned}
$$

Thus U is already diagonal and we don't need to go through steps 2 and 3, the scaling and second orthogonalization, except to note that if we skip the scaling the angular frequencies are given by $\omega_i^2 =$ coefficient in U / coefficient in T. Thus we have one normal mode, x_1, with $\omega_1 = \sqrt{k/m_O}$, with $x_2 = 0$, $\eta_1 = -\eta_2$, $q_3 = 0$, in which the two oxygens vibrate in and out together, symmetrically about the carbon, which doesn't move. We also have another mode, x_2, with

$$
\omega_2 = \sqrt{\frac{k(m_C+2m_O)^2/m_C^2}{m_O(1+2m_O/m_C)}} = \sqrt{\frac{k(m_C+2m_O)}{m_Om_C}},
$$

with $x_1 = 0$, $\eta_1 = \eta_2$, in which the two oxygens move right or left together, with the carbon moving in the opposite direction.

We have successfully solved for the longitudinal vibrations by eliminating one of the degrees of freedom. Let us now try the second method, in which we choose an arbitrary equilibrium position $q_1 = -b$, $q_2 = b$, $q_3 = 0$. Then

$$
\begin{aligned}
T &= \frac{1}{2}m_O(\dot{\eta}_1^2+\dot{\eta}_2^2) + \frac{1}{2}m_C\dot{\eta}_3^2 \\
U &= \frac{1}{2}k\left[(\eta_1-\eta_3)^2+(\eta_2-\eta_3)^2\right].
\end{aligned}
$$

T is already diagonal, so $\mathcal{O}_1 = \mathbb{1}$, $x = \eta$. In the second step S is the diagonal matrix with $S_{11} = S_{22} = \sqrt{m_O}$, $S_{33} = \sqrt{m_C}$, and $y_i = \sqrt{m_O}\eta_i$ for $i = 1, 2$,

and $y_3 = \sqrt{m_C}\eta_3$. Then

$$\begin{aligned} U &= \frac{1}{2}k\left[\left(\frac{y_1}{\sqrt{m_O}} - \frac{y_3}{\sqrt{m_C}}\right)^2 + \left(\frac{y_2}{\sqrt{m_O}} - \frac{y_3}{\sqrt{m_C}}\right)^2\right] \\ &= \frac{1}{2}\frac{k}{m_O m_C}\left[m_C y_1^2 + m_C y_2^2 + 2m_O y_3^2 - 2\sqrt{m_O m_C}(y_1+y_2)y_3\right]. \end{aligned}$$

Thus the matrix B is

$$B = \begin{pmatrix} m_C & 0 & -\sqrt{m_O m_C} \\ 0 & m_C & -\sqrt{m_O m_C} \\ -\sqrt{m_O m_C} & -\sqrt{m_O m_C} & 2m_O \end{pmatrix},$$

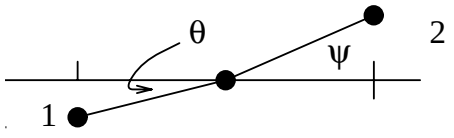
which is singular, as it annihilates the vector $y^T = (\sqrt{m_O}, \sqrt{m_O}, \sqrt{m_C})$, which corresponds to $\eta^T = (1,1,1)$, *i.e.* all the nuclei are moving by the same amount, or the molecule is translating rigidly. Thus this vector corresponds to a zero eigenvalue of U, and a harmonic oscillation of zero frequency. This is free motion[3], $\eta = \eta_0 + vt$. The other two modes can be found by diagonalizing the matrix, and will be as we found by the other method.

Transverse motion

What about the transverse motion? Consider the equilibrium position of the molecule to lie in the x direction, and consider small deviations in the z direction. The kinetic energy

$$T = \frac{1}{2}m_O\dot{z}_1 + \frac{1}{2}m_O\dot{z}_2^2 + \frac{1}{2}m_C\dot{z}_3^2.$$

is already diagonal, just as for the longitudinal modes in the second method. Any potential energy must be due to a resistance to bending, so to second order, $U \propto (\psi-\theta)^2 \sim (\tan\psi - \tan\theta)^2 = [(z_2-z_3)/b + (z_1-z_3)/b]^2 = b^{-2}(z_1 + z_2 - 2z_3)^2$.



Note that the potential energy is proportional to the square of a single linear

[3]To see that linear motion is a limiting case of harmonic motion as $\omega \to 0$, we need to choose the complex coefficient to be a function of ω, $A(\omega) = x_0 - iv_0/\omega$, with x_0 and v_0 real. Then $x(t) = \lim_{\omega\to 0}\ \mathrm{Re}\ A(\omega)e^{i\omega t} = x_0 + v_0 \lim_{\omega\to 0}\sin(\omega t)/\omega = x_0 + v_0 t$

combination of the displacements, or to the square of one component (with respect to a particular direction) of the displacement. Therefore there is no contribution of the two orthogonal directions, and there are two zero modes, or two degrees of freedom with no restoring force. One of these is the center of mass motion, $z_1 = z_2 = z_3$, and the other is the third direction in the abstract space of possible displacements, $z^T = (1, -1, 0)$, with $z_1 = -z_2$, $z_3 = 0$, which we see is a rotation. Thus there remains only one true transverse vibrational mode in the z direction, and also one in the y direction, which together with the two longitudinal ones we found earlier, make up the 4 vibrational modes we expected from the general formula $2(n-2)$ for a collinear molecule.

You might ask whether these oscillations we have discussed are in any way observable. Quantum mechanically, a harmonic oscillator can only be in states with excitation energy $E = n\hbar\omega$, where $n \in \mathbb{Z}$ is an integer and $2\pi\hbar$ is Planck's constant. When molecules are in an excited state, they can emit a photon while changing to a lower energy state. The energy of the photon, which is the amount lost by the molecule, is proportional to the frequency, $\Delta E = 2\pi\hbar f$, so by measuring the wavelength of the emitted light, we can determine the vibrational frequencies of the molecules. So the calculations we have done, and many others for which we have built the apparatus, are in fact very practical tools for molecular physics.

5.1.2 An Alternative Approach

The step by step diagonalization we just gave is not the easiest approach to solving the linear differential equation (5.3). Solutions to linear differential equations are subject to superposition, and equations with coefficients independent of time are simplified by Fourier transform, so we can express the N dimensional vector of functions $\eta_j(t)$ as

$$\eta_j(t) = \int_{-\infty}^{\infty} d\omega \psi_j(\omega) e^{-i\omega t}.$$

Then the Lagrange equations become

$$\int_{-\infty}^{\infty} d\omega \sum_j \left(A_{ij} - \omega^2 M_{ij}\right) \psi_j(\omega) e^{-i\omega t} = 0 \quad \text{for all } t.$$

But $e^{-i\omega t}$ are linearly independent functions of $t \in \mathbb{R}$, so

$$\sum_j \left(A_{ij} - \omega^2 M_{ij}\right) \psi_j(\omega) = 0.$$

This implies $\psi_j(\omega) = 0$ except when the matrix $A_{ij} - \omega^2 M_{ij}$ is singular, $\det(A_{ij} - \omega^2 M_{ij}) = 0$, which gives a discrete set of angular frequencies $\omega_1 \ldots \omega_N$, and for each ω_j an eigenvector ψ_j.

5.2 Other interactions

In our treatment we assumed a Lagrangian formulation with a kinetic term purely quadratic in $\dot{\vec{q}}$, together with a velocity independent potential. There is a wider scope of small oscillation problems which might include dissipative forces like friction, or external time-dependent forces, or perhaps terms in the Lagrangian linear in the velocities. An example of the latter occurs in rotating reference frames, from the Coriolus force, and is important in the question of whether there is a gravitationally stable location for small objects caught between the Earth and the moon at the "L5" point[4]. Each of these complications introduces terms, even in the linear approximation to the equations of motion, which cannot be diagonalized away, because there is not significant freedom of diagonalization left, in general, after having simplified T and U. Thus the approach of section 5.1 does not generalize well, but the approach of section 5.1.2 can be applied.

For example, we might consider adding a generalized force Q_i on η_i, consisting of a dissipative force $\sum_j R_{ij}\dot{\eta}_j$ and a driving force F_i. We will assume R is a symmetric matrix, as might be a result of a Rayleigh dissipation function (see Section 2.7 or Ref. [6]). We will consider the motion to first order in F, so any coordinate dependence of R or F is replaced, as it was for M and A, by their values at the equilibrium position. Thus the equations of motion become

$$\sum_j \left(M_{ij}\ddot{\eta}_j + R_{ij}\dot{\eta}_j + A_{ij}\eta_j\right) - F_i = 0.$$

Again making the ansatz that

$$\eta_j(t) = \int_{-\infty}^{\infty} d\omega \psi_j(\omega) e^{-i\omega t}$$

and expressing $F_i(t)$ in terms of its fourier transform

$$F_j(t) = \int_{-\infty}^{\infty} d\omega \tilde{f}_i(\omega) e^{-i\omega t}$$

[4]See problem 5.3.

we find

$$\sum_j \left(-\omega^2 M_{ij} - i\omega R_{ij} + A_{ij}\right)\psi_j = \tilde{f}_i.$$

Except for at most $2N$ values of ω the matrix multiplying ψ_j will have a non-zero determinant and will be invertible, allowing us to find the response ψ_j to the fourier component of the driving force, $\tilde{f}_i$. Those values of ω for which the determinant vanishes, and the vector ψ_j which the matrix annihilates, correspond to damped modes that we would see if the driving force were removed.

5.2.1 Forced Harmonic Oscillations

In this section we will explore the effects of driving forces on oscillators. For simplicity let us consider a damped oscillator with one degree of freedom, with a driving force $F(t)$:

$$m\ddot{x}(t) + R\dot{x}(t) + kx(t) = F(t).$$

For this linear oscillator, we can solve by Fourier transform. Writing $x(t) = \int_{-\infty}^{\infty} \tilde{x}(\omega)e^{-i\omega t}\,d\omega$, we find $\left(-m\omega^2 - iR\omega + k\right)\tilde{x}(\omega) = \tilde{F}(\omega)$, where the Fourier transformed force is $\tilde{F}(\omega) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(t)e^{i\omega t}dt$.

Without any forces, we have solutions for $\omega^2 + 2i\rho\omega - \omega_0^2 = 0$, (where $\omega_0 := \sqrt{k/m}$, $\rho = R/2m$), so the solutions are at $\omega = -i\rho \pm \bar{\omega}$, with $\bar{\omega} = \sqrt{\omega_0^2 - \rho^2}$. Due to the negative imaginary part of either of these ωs, the unforced oscillations will decay with time. If we do have a forcing function, however, we have an inhomogeneous solution (with $\tilde{f} = \tilde{F}/m$)

$$\tilde{x}(\omega) = \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 - 2i\rho\omega}.$$

As $x(t)$ and $f(t)$ are real-valued functions of time, the fourier transforms must satisfy $\tilde{x}^*(\omega) = \tilde{x}(-\omega)$, $\tilde{f}^*(\omega) = \tilde{f}(-\omega)$, and the steady state solution is

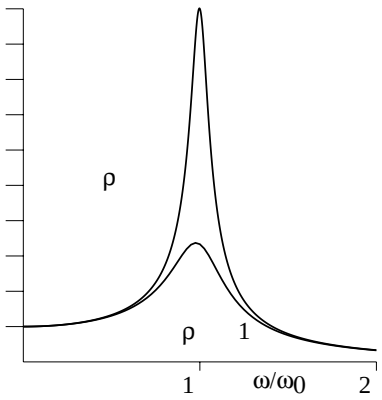
$$\begin{aligned} x(t) &= \int_0^{\infty} d\omega \left(\tilde{x}(\omega)e^{-i\omega t} + \tilde{x}^*(\omega)e^{i\omega t}\right) = 2\ \mathrm{Re}\ \int_0^{\infty} d\omega \tilde{x}(\omega)e^{-i\omega t} \\ &= 2\ \mathrm{Re}\ \int_0^{\infty} d\omega \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 - 2i\rho\omega}e^{-i\omega t}. \end{aligned} \tag{5.4}$$

If we consider a forcing function of only one positive frequency, say $\tilde{f}(\omega) = a\delta(\omega - \omega_{\rm ex})$ for $\omega \geq 0$, we have

$$x(t) = 2 \text{ Re } \frac{a}{\omega_0^2 - \omega_{\rm ex}^2 - 2i\rho\omega_{\rm ex}} e^{-i\omega_{\rm ex}t},$$

with amplitude

$$\begin{aligned} A &= \left| \frac{2a}{\omega_0^2 - \omega_{\rm ex}^2 - 2i\rho\omega_{\rm ex}} \right| \\ &= \frac{2|a|}{\sqrt{\left(\omega_0^2 - \omega_{\rm ex}^2\right)^2 + 4\rho^2\omega_{\rm ex}^2}}. \end{aligned}$$



We see that the response in the frequency domain is proportional to the force, with a frequency dependence which is sharply peaked if the damping coefficient is small compared to the natural frequency, $\rho \ll \omega_0$.

If we ask in the temporal domain, what is the effect on $x(t)$ of a force $f(t')$, we have

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} d\omega \, \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \, \frac{f(t')}{\omega_0^2 - \omega^2 - 2i\rho\omega} e^{i\omega(t-t')} \\ &= \int_{-\infty}^{\infty} dt' \, G(t-t') \frac{f(t')}{m}, \end{aligned} \tag{5.5}$$

where

$$G(t-t') := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-t')}}{\omega_0^2 - \omega^2 - 2i\rho\omega} \tag{5.6}$$

is the temporal Green's function. Note that it would appear from (5.5) that effects could preceed causes, as the integral is over all t', including times after t, but in fact G vanishes there. For $t' > t$ we may evaluate (5.6) by closing the integration contour in the upper half plane, for the exponential will vanish for large $\text{Im } \omega > 0$ when $t < t'$. As the integrand is analytic in the upper half plane, the contour integral vanishes, and $G(t-t') = 0$ for $t' > t$. On the other hand, for $\Delta t = t - t' > 0$, the integration contour may be closed in the lower half plane, picking up the residues from the poles at

$\omega = -i\rho \pm \bar{\omega}$. The residue there is $\mp e^{-\rho\,\Delta t} e^{\mp i\bar{\omega}\Delta t}/\bar{\omega}$, so

$$G(\Delta t) = \frac{1}{\bar{\omega}} e^{-\rho\,\Delta t} \sin\bar{\omega}\Delta t.$$

Weakly nonlinear oscillating systems

The oscilator we just considered could be solved exactly because it is a linear system. The equation of motion is a linear operator (including time derivative operators) acting on the dynamical variable $x(t)$, set equal to a forcing term which is a given function of time. Most systems, however, are not exactly linear. If the equation of motion is close to linear, we might imagine a perturbative calculation in which we bring the difference from linearity, considered small, to the right hand side, evaluate it in the linear approximation, and consider it a forcing term. For example, we are quite used to the idea that a pendulum may be approximated by a harmonic oscillator. A forced, linearly damped pendulum has an equation of motion

$$m\ell^2\ddot{\theta} + R\dot{\theta} + mg\ell\sin\theta = F(t),$$

which in the approximation $\sin\theta \approx \theta$ reduces to the harmonic oscillator we just considered. More precisely, we can write

$$\ddot{\theta} + 2\rho\dot{\theta} + \omega_0^2\theta = f(t) - \omega_0^2(\sin\theta - \theta),$$

where $\rho = R/2m\ell^2$, $\omega_0 = \sqrt{g/\ell}$ and $f(t) = F(t)/m\ell^2$. If the forcing function $f(t)$ and the oscillations are small ($\theta \ll \pi$), we can imagine a sequence of approximations, first evaluating $\theta(t)$ dropping the $(\sin\theta - \theta)$ term, and then evaluating the $n+1$'st approximation to $\theta(t)$ by using the n'th approximation to evaluate $(\sin\theta - \theta)(t)$ as a forcing term.

We will return to this issue, discussing both how this works and why it may not be the ideal way to do a perturbative expansion, in Chapter 7.

5.3 String dynamics

In this section we consider two closely related problems, transverse oscillations of a stretched loaded string, and of a stretched heavy string. The latter is is a limiting case of the former. This will provide an introduction to field theory, in which the dynamical degrees of freedom are not a discrete set but

are defined at each point in space. In Chapter 8 we will discuss more interesting and involved cases such as the electromagnetic field, where at each point in space we have $\vec{E}$ and $\vec{B}$ as degrees of freedom, though not without constraints.

The loaded string we will consider is a light string under tension τ stretched between two fixed points a distance ℓ apart, say at $x = 0$ and $x = \ell$. On the string, at points $x = a, 2a, 3a, \ldots, na$, are fixed n particles each of mass m, with the first and last a distance a away from the fixed ends. Thus $\ell = (n+1)a$. We will consider only small transverse motion of these masses, using y_i as the transverse displacement of the i'th mass, which is at $x = ia$. We assume all excursions from the equilibrium positions $y_i = 0$ are small, and in particular that the difference in successive displacements $y_{i+1} - y_i \ll a$. Thus we are assuming that the angle made by each segment of the string, $\theta_i = \tan^{-1}[(y_{i+1} - y_i)/a] \ll 1$. Working to first order in the θ's in the equations of motion, and second order for the Lagrangian, we see that restricting our attention to transverse motions and requiring no horizontal motion forces taking the tension τ to be constant along the string. The transverse force on the i'th mass is thus

$$F_i = \tau \frac{y_{i+1} - y_i}{a} + \tau \frac{y_{i-1} - y_i}{a} = \frac{\tau}{a}(y_{i+1} - 2y_i + y_{i-1}).$$

The potential energy $U(y_1, \ldots, y_n)$ then satisfies

$$\frac{\partial U}{\partial y_i} = -\frac{\tau}{a}(y_{i+1} - 2y_i + y_{i-1})$$

so

$$\begin{aligned} &U(y_1, \ldots, y_i, \ldots, y_n) \\ &\quad = \int_0^{y_i} dy_i \, \frac{\tau}{a}(2y_i - y_{i+1} - y_{i-1}) + F(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \\ &\quad = \frac{\tau}{a}\left(y_i^2 - (y_{i+1} + y_{i-1})y_i\right) + F(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \\ &\quad = \frac{\tau}{2a}\left((y_{i+1} - y_i)^2 + (y_i - y_{i-1})^2\right) + F'(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \\ &\quad = \sum_{i=0}^{n} \frac{\tau}{2a}(y_{i+1} - y_i)^2 + \text{constant}. \end{aligned}$$

The F and F' are unspecified functions of all the y_j's except y_i. In the last expression we satisfied the condition for all i, and we have used the convenient definition $y_0 = y_{n+1} = 0$. We can and will drop the arbitrary constant.

The kinetic energy is $T = \frac{1}{2}m\sum_1^n \dot{y}_i^2$.

Before we continue with the analysis of this problem, let us note that another physical setup also leads to the same Lagrangian. Consider a one dimensional lattice of identical atoms with a stable equilibrium in which they are evenly spaced, with interactions between nearest neighbors. Let η_i be the longitudinal displacement of the i'th atom from its equilibrium position. The kinetic energy is simply $T = \frac{1}{2}m\sum_1^n \dot{\eta}_i^2$. As the interatomic distance differs from its equilibrium position by $\eta_{i+1} - \eta_i$, the interaction potential of atoms i and $i+1$ can be approximated by $U(\eta_{i+1}, \eta_i) \approx \frac{1}{2}k(\eta_{i+1} - \eta_i)^2$. We have in effect atoms separated by springs of spring constant k, and we see that if $k = \tau/a$, we get the same Lagrangian for longitudinal oscillations of this lattice as we had for the transverse oscillations of the loaded string.

As the kinetic energy is simply $T = \frac{1}{2}m\sum_1^n \dot{y}_i^2$, the mass matrix is already proportional to the identity matrix and we do not need to go through the first two steps of our general process. The potential energy $U = \frac{1}{2}y^T \cdot A \cdot y$ has a non-diagonal $n \times n$ matrix

$$A = -\frac{\tau}{a}\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}.$$

Diagonalizing even a 3×3 matrix is work, so an $n \times n$ matrix might seem out of the question, without some hints from the physics of the situation. In this case the hint comes in a roundabout fashion — we will first consider a limit in which $n \to \infty$, the **continuum limit**, which leads to an interesting physical situation in its own right.

Suppose we consider the loaded string problem in the limit that the spacing a becomes very small, but the number of masses m becomes large, keeping the total length ℓ of the string fixed. If at the same time we adjust the individual masses so that the mass per unit length, ρ, is fixed, our bumpy string gets smoothed out in the limit, and we might expect that in this limit we reproduce the physical problem of transverse modes of a uniformly dense stretched string, like a violin string. Thus we wish to consider the limit

$$a \to 0, \quad n \to \infty, \quad \ell = (n+1)a \text{ fixed}, \quad m \to 0, \quad \rho = m/a \text{ fixed}.$$

It is natural to think of the degrees of freedom as associated with the label x rather than i, so we redefine the dynamical functions $\{y_j(t)\}$ as $y(x,t)$, with $y(ja,t) = y_j(t)$. While this only defines the function at discrete points in x, these are closely spaced for small a and become dense as $a \to 0$. We will assume that the function $y(x)$ is twice differentiable in the continuum limit, though we shall see that this is not the case for all possible motions of the discrete system.

What happens to the kinetic and potential energies in this limit? For the kinetic energy,

$$T = \frac{1}{2}m\sum_i \dot{y}_i^2 = \frac{1}{2}\rho\sum_i a\dot{y}^2(x_i) = \frac{1}{2}\rho\sum_i \Delta x\dot{y}^2(x_i) \to \frac{1}{2}\rho\int_0^\ell dx\,\dot{y}^2(x),$$

where the next to last expression is just the definition of a Riemann integral. For the potential energy,

$$U = \frac{\tau}{2a}\sum_i (y_{i+1}-y_i)^2 = \frac{\tau}{2}\sum_i \Delta x\left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2 \to \frac{\tau}{2}\int_0^\ell dx\,\left(\frac{\partial y}{\partial x}\right)^2.$$

The equation of motion for y_i is

$$m\ddot{y}_i = \frac{\partial L}{\partial y_i} = -\frac{\partial U}{\partial y_i} = \frac{\tau}{a}[(y_{i+1}-y_i)-(y_i-y_{i-1})],$$

or

$$\rho a\ddot{y}(x) = \frac{\tau}{a}([y(x+a)-y(x)]-[y(x)-y_(x-a)]).$$

We need to be careful about taking the limit

$$\frac{y(x+a)-y(x)}{a} \longrightarrow \frac{\partial y}{\partial x}$$

because we are subtracting two such expressions evaluated at nearby points, and because we will need to divide by a again to get an equation between finite quantities. Thus we note that

$$\frac{y(x+a)-y(x)}{a} = \left.\frac{\partial y}{\partial x}\right|_{x+a/2} + \mathcal{O}(a^2),$$

so

$$\begin{aligned}\rho\ddot{y}(x) &= \frac{\tau}{a}\left(\frac{y(x+a)-y(x)}{a} - \frac{y(x)-y_(x-a)}{a}\right)\\ &\approx \frac{\tau}{a}\left(\left.\frac{\partial y}{\partial x}\right|_{x+a/2} - \left.\frac{\partial y}{\partial x}\right|_{x-a/2}\right) \to \tau\frac{\partial^2 y}{\partial x^2},\end{aligned}$$

and we wind up with the wave equation for transverse waves on a massive string

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0,$$

where

$$c = \sqrt{\frac{\tau}{\rho}}.$$

Solving this wave equation is very simple. For the fixed boundary conditions $y(x) = 0$ at $x = 0$ and $x = \ell$, the solution is a fourier expansion

$$y(x,t) = \sum_{p=1}^{\infty} \text{Re } B_p e^{ick_p t} \sin k_p x,$$

where $k_p \ell = p\pi$. Each p represents one normal mode, and there are an infinite number as we would expect because in the continuum limit there are an infinite number of degrees of freedom.

We have certainly not shown that $y(x) = B \sin kx$ is a normal mode for the problem with finite n, but it is worth checking it out. This corresponds to a mode with $y_j = B \sin kaj$, on which we apply the matrix A

$$\begin{aligned}
(A \cdot y)_i &= \sum_j A_{ij} y_j = -\frac{\tau}{a} \left(y_{i+1} - 2y_i + y_{i-1} \right) \\
&= -\frac{\tau}{a} B \left(\sin(kai + ka) - 2\sin(kai) + \sin(kai - ka) \right) \\
&= -\frac{\tau}{a} B (\sin(kai)\cos(ka) + \cos(kai)\sin(ka) - 2\sin(kai) \\
&\qquad + \sin(kai)\cos(ka) - \cos(kai)\sin(ka)) \\
&= \frac{\tau}{a} B \left(2 - 2\cos(ka) \right) \sin(kai) \\
&= \frac{2\tau}{a} \left(1 - \cos(ka) \right) y_i.
\end{aligned}$$

So we see that it *is* a normal mode, although the frequency of oscillation

$$\omega = \sqrt{\frac{2\tau}{am}(1 - \cos(ka))} = 2\sqrt{\frac{\tau}{\rho}} \frac{\sin(ka/2)}{a}$$

differs from $k\sqrt{\tau/\rho}$ except in the limit $a \to 0$ for fixed k.

The **wave numbers** k which index the normal modes are restricted by the fixed ends to the discrete set $k = p\pi/\ell = p\pi/(n+1)a$, for $p \in \mathbb{Z}$, *i.e.* p is

an integer. This is still too many (∞) for a system with a finite number of degrees of freedom. The resolution of this paradox is that not all different k's correspond to different modes. For example, if $p' = p + 2m(n+1)$ for some integer m, then $k' = k + 2\pi m/a$, and $\sin(k'aj) = \sin(kaj + 2m\pi) = \sin(kaj)$, so k and k' represent the same normal mode. Also, if $p' = 2(n+1) - p$, $k' = (2\pi/a) - k$, $\sin(k'aj) = \sin(2\pi - kaj) = -\sin(kaj)$, so k and k' represent the same normal mode, with opposite phase. Finally $p = n+1$, $k = \pi/a$ gives $y_j = B\sin(kaj) = 0$ for all j and is not a normal mode. This leaves as independent only $p = 1, ..., n$, the right number of normal modes for a system with n degrees of freedom.

The angular frequency of the p'th normal mode

$$\omega_p = 2\sqrt{\frac{\tau}{ma}}\sin\frac{p\pi}{2(n+1)}$$

in plotted in Fig. 5.3. For fixed values of p and ρ, as $n \to \infty$,

$$\omega_p = 2\sqrt{\frac{\tau}{\rho}}\frac{1}{a}\sin\frac{pa\pi}{2\ell} \to 2\sqrt{\frac{\tau}{\rho}}\frac{p\pi}{2\ell} = ck_p,$$

as we have in the continuum limit. But if we consider modes with a fixed ratio of p/n as $n \to \infty$, we do not have a smooth limit $y(x)$, and such nodes are not appropriate for the continuum limit. In the physics of crystals, the former kind of modes are known as **acoustic modes**, while the later modes, in particular those for $n - p$ fixed, which depend on the discrete nature of the crystal, are called **optical modes**.

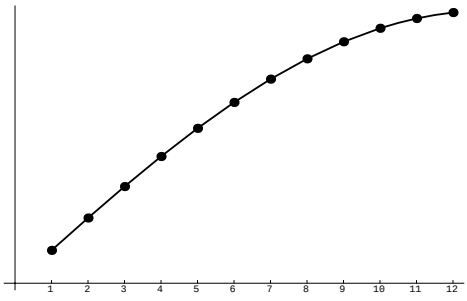


Fig. 5.3. Frequencies of oscillation of the loaded string.

5.4 Field theory

We now examine how to formulate the continuum limit directly.

5.4.1 Lagrangian density

We saw in the last section that the kinetic and potential energies in the continuum limit can be written as integrals over x of densities, and so we may also write the Lagrangian as the integral of a **Lagrangian density** $\mathcal{L}(x)$,

$$L = T - U = \int_0^L dx \mathcal{L}(x), \qquad \mathcal{L}(x) = \left[\frac{1}{2}\rho \dot{y}^2(x,t) - \frac{1}{2}\tau \left(\frac{\partial y(x,t)}{\partial x} \right)^2 \right].$$

This Lagrangian, however, will not be of much use until we figure out what is meant by varying it with respect to each dynamical degree of freedom or its corresponding velocity. In the discrete case we have the canonical momenta $P_i = \partial L/\partial \dot{y}_i$, where the derivative requires holding all $\dot{y}_j$ fixed, for $j \neq i$, as well as all y_k fixed. This extracts one term from the sum $\frac{1}{2}\rho \sum a \dot{y}_i^2$, and this would appear to vanish in the limit $a \to 0$. Instead, we define the canonical momentum as a density, $P_i \to aP(x = ia)$, so

$$P(x = ia) = \lim \frac{1}{a} \frac{\partial}{\partial \dot{y}_i} \sum_i a \, \mathcal{L}(y(x), \dot{y}(x), x)|_{x=ai} \, .$$

We may think of the last part of this limit,

$$\lim_{a\to 0} \sum_i a \, \mathcal{L}(y(x), \dot{y}(x), x)|_{x=ai} = \int dx \, \mathcal{L}(y(x), \dot{y}(x), x),$$

if we also define a limiting operation

$$\lim_{a\to 0} \frac{1}{a} \frac{\partial}{\partial \dot{y}_i} \to \frac{\delta}{\delta \dot{y}(x)},$$

and similarly for $\frac{1}{a}\frac{\partial}{\partial y_i}$, which act on functionals of $y(x)$ and $\dot{y}(x)$ by

$$\frac{\delta y(x_1)}{\delta y(x_2)} = \delta(x_1 - x_2), \quad \frac{\delta \dot{y}(x_1)}{\delta y(x_2)} = \frac{\delta y(x_1)}{\delta \dot{y}(x_2)} = 0, \quad \frac{\delta \dot{y}(x_1)}{\delta \dot{y}(x_2)} = \delta(x_1 - x_2).$$

Here $\delta(x' - x)$ is the **Dirac delta function**, defined by its integral,

$$\int_{x_1}^{x_2} f(x')\delta(x' - x)dx' = f(x)$$

for any function $f(x)$, provided $x \in (x_1, x_2)$. Thus

$$P(x) = \frac{\delta}{\delta \dot{y}(x)} \int_0^\ell dx' \frac{1}{2} \rho \dot{y}^2(x', t) = \int_0^\ell dx' \rho \dot{y}(x', t) \delta(x' - x) = \rho \dot{y}(x, t).$$

We also need to evaluate

$$\frac{\delta}{\delta y(x)} L = \frac{\delta}{\delta y(x)} \int_0^\ell dx' \frac{-\tau}{2} \left(\frac{\partial y}{\partial x} \right)^2_{x=x'}.$$

For this we need

$$\frac{\delta}{\delta y(x)} \frac{\partial y(x')}{\partial x'} = \frac{\partial}{\partial x'} \delta(x' - x) := \delta'(x' - x),$$

which is again defined by its integral,

$$\begin{aligned} \int_{x_1}^{x_2} f(x') \delta'(x' - x) dx' &= \int_{x_1}^{x_2} f(x') \frac{\partial}{\partial x'} \delta(x' - x) dx' \\ &= f(x') \delta(x' - x) |_{x_1}^{x_2} - \int_{x_1}^{x_2} dx' \, \frac{\partial f}{\partial x'} \delta(x' - x) \\ &= \frac{\partial f}{\partial x}(x), \end{aligned}$$

where after integration by parts the surface term is dropped because $\delta(x - x') = 0$ for $x \neq x'$, which it is for $x' = x_1, x_2$ if $x \in (x_1, x_2)$. Thus

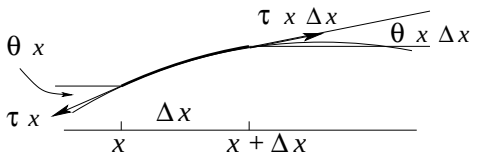
$$\frac{\delta}{\delta y(x)} L = - \int_0^\ell dx' \tau \frac{\partial y}{\partial x}(x') \delta'(x' - x) = \tau \frac{\partial^2 y}{\partial x^2},$$

and Lagrange's equations give the wave equation

$$\rho \ddot{y}(x, t) - \tau \frac{\partial^2 y}{\partial x^2} = 0. \tag{5.7}$$

We have derived the wave equation for small transverse deformations of a stretched string by considering the continuum limit of a loaded string, in the process demonstating how to formulate Lagrangian mechanics for a continuum system. Of course it is more usual, and simpler, to derive it directly by considering Newton's law on an infinitesimal element of the string. Let's include gravity for good measure. If the string point initially at x

has a transverse displacement $y(x)$ and a longitudinal displacement $\eta(x)$, both considered small, the slope of the string dy/dx is also small. The segment $[x, x+\Delta x]$ has a mass $\rho\Delta x$, where as before ρ is the mass per unit length, and the forces on it are



in x direction: $\tau(x+\Delta x)\cos\theta(x+\Delta x) - \tau(x)\cos\theta(x) = \rho\Delta x\ddot{\eta}$

in y direction: $\tau(x+\Delta x)\sin\theta(x+\Delta x) - \tau(x)\sin\theta(x) - \rho g\Delta x = \rho\Delta x\ddot{y}$

As $\theta << 1$, we can replace $\cos\theta$ by 1 and $\sin\theta$ with $\tan\theta = \partial y/\partial x$, and then from the first equation we see that $\partial\tau/\partial x$ is already small, so we can consider τ as constant in the second equation, which gives

$$\tau\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) - \rho g\Delta x = \rho\Delta x\ddot{y},$$

or

$$\tau\frac{\partial^2 y}{\partial x^2} - \rho g = \rho\ddot{y}.$$

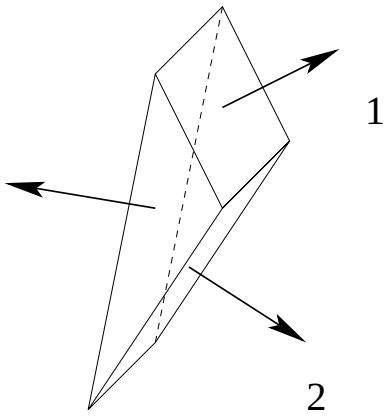
This agrees with Eq. 5.7 if we drop the gravity term, which we had not included in our discussion of the loaded string.

5.4.2 Three dimensional continua

Could we do the same kind of analysis on a three dimensional solid object? We might label each piece of the object with an equilibrium or reference position $\vec{x}$, and consider the dynamics of possible displacements $\vec{\eta}(\vec{x})$. We will assume this displacement is small and smooth function of $\vec{x}$ and t, in fact twice differentiable. Consider the dynamics of an infinitesimal volume element ΔV. The acceleration of each volume element will be determined by the ratio of the net force on that volume to its mass, $\rho\Delta V$, where ρ is now the density, mass per unit volume, and is also assumed to be a smooth function, though not necessarily constant. The forces we will consider will be of two types. There may be external forces which will be taken to be extensive, that is, proportional to the volume, called **volume forces**. One example is gravity near the Earth's surface, with $\vec{F} = -\rho g\Delta V\hat{e}_z$. If the material under

discussion had an electric charge density $\rho_E(\vec{x})$ in an external electric field $\vec{E}(\vec{x})$, there would also be a volume force $\rho_E(\vec{x})\vec{E}(\vec{x})\,\Delta V$. In addition to the volume forces, there are also **surface force**s which the rest of the object exerts on the element under consideration. We will assume that all such forces are local, due to the material on the other side of the surface, and continuously varying, so the force across an infinitesimal element of surface dS will be proportional to its area, at least if we keep the direction fixed.

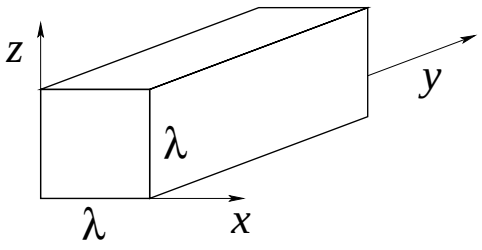
In fact, we can show that the force across an infinitesimal surface $d\vec{S}$ is linear in the vector $d\vec{S}$ even when the direction changes. Consider two elements $d\vec{S}_1$ and $d\vec{S}_2$, shown as rectangles, and the third side $d\vec{S}_3$, which is the opposite of their sum in the limit that size shrinks to zero. Together with the two parallel triangular pieces, these bound an infinitesimal volume. Let us scale the whole picture by a factor λ. The force on each side is proportional to λ^2, but the mass of the volume is proportional to λ^3, so as $\lambda \to 0$, the



coefficient of λ^2 in the sum of the forces must vanish. The triangular pieces cancel each other, so the sum of the forces through $d\vec{S}_1$ and $d\vec{S}_2$ cancels the force through $d\vec{S}_3$. That is, the force is linear (additive) in the surface elements $d\vec{S}$.

But the force is not necessarily in the same direction as $d\vec{S}$. This would be true for the pressure in a gas, or in a nonviscous or static fluid, in which no tangential forces could be exerted along the boundary. But more generally, a force linear in $d\vec{S}$ will be specified by a matrix, and the force exerted on dV across $d\vec{S}$ will be $F_i = -\sum_j \mathbf{P}_{ij} dS_j$, where $\mathbf{P}$ is known as the **stress tensor**[5].

[5]To be clear: $\sum_j \mathbf{P}_{ij} dS_j$ is the force exerted by the back side of the surface element on the front side, so if $d\vec{S}$ is an outward normal, the force *on* the volume is $-\int_S \sum_j \mathbf{P}_{ij} dS_j$, and a pressure corresponds to $\mathbf{P} = +p\delta_{ij}$. This agrees with Symon ([17]) but has a reversed sign from Taylor's ([18]) $\mathbf{\Sigma} = -\mathbf{P}$.



Though $\mathbf{P}$ is not a scalar or diagonal in general, there is one constraint on the stress tensor — it is symmetric. To see this, consider the prism shown, and the torque in the y direction. The forces across the two faces perpendicular to z are of order λ, and are equal and opposite, so they provide a torque $-\lambda^2 h\mathbf{P}_{xz}$ in the y direction. Similarly the two faces perpendicular to x provide a torque $+\lambda^2 h\mathbf{P}_{zx}$ in that direction. The equal forces on other two faces have a moment arm parallel to y and therefore provide no torque in that direction. But the moment of inertia about the y axis is of order $\lambda^2 dV = \lambda^4 h$. So if the angular acceleration is to remain finite as $\lambda \to 0$, we must have $\mathbf{P}_{zx} - \mathbf{P}_{xz} = 0$, and $\mathbf{P}$ must be a symmetric matrix.

We expect that the stress forces the material on one side of a boundary exerts on the other is due to some distortion of the material. Near any value of x, we may expand the displacement as

$$\eta_i(x + \Delta x) = \eta_i(x) + \sum_j \frac{\partial \eta_i}{\partial x_j}\Delta x_j + ...$$

Moving the entire object as a whole, $\vec{\eta}(x) =$ constant, or rotating it as a rigid body about an axis $\vec{\omega}$, with $\partial\eta_i/\partial x_j = \sum_k \epsilon_{ijk}\omega_k$, will not produce any stress, and so we will not consider such displacements to be part of the **strain tensor**, which we therefore define to be the symmetric part of the derivative matrix:

$$\mathbf{S}_{ij} = \frac{1}{2}\left(\frac{\partial \eta_i}{\partial x_j} + \frac{\partial \eta_j}{\partial x_i}\right).$$

In general, the properties of the material will determine how the stress tensor is related to the strain tensor, though for small displacements we expect it to depend linearly.

Even linear dependence could be quite complex, but if the material properties are rotationally symmetric, things are fairly simple. Of course in a crystal we might not satisfy that condition, but if we do assume the functional dependence of the stress on the strain is rotationally invariant, we may find the most general possibilities by decomposing the tensors into pieces which

behave suitably under rotations. Here we are generalizing the idea that a vector cannot be defined in terms of pure scalars, and a scalar can depend on vectors only through a scalar product. A symmetric tensor consists of a piece, its trace, which behaves like a scalar, and a traceless piece, called the **deviatoric part**, which behaves differently, as an irreducible representation[6]. The only possible linear relationships are thus

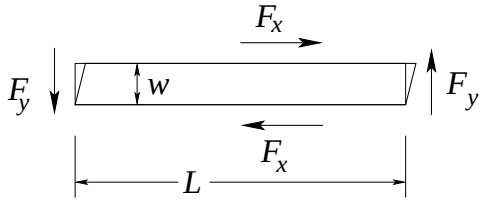
$$\mathrm{Tr}\,\mathbf{P} = -\alpha\,\mathrm{Tr}\,\mathbf{S}; \qquad \mathbf{P}_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{Tr}\,\mathbf{P} = -\beta\left(\mathbf{S}_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{Tr}\,\mathbf{S}\right). \tag{5.8}$$

These are known as the **generalized Hooke's law** for an elastic solid.

The tensor stress and strain we have described here are perhaps not as familiar as some other relations met in more elementary courses. First consider the **bulk modulus** B, the inverse of the ratio of the fractional decrease in volume to the isotropic pressure which causes it. Here the stress and strain tensors are both multiples of the identity, $\mathbf{P} = +p\delta_{ij}$ and $d\vec{\eta} = -cd\vec{x}$, so $\mathbf{S} = -c\delta_{ij}$ and $c = p/\alpha$. For a linear contraction $\vec{x} \to \vec{x} - c\vec{x}$ the volume will contract by $\Delta V = -3cV$. Therefore the

$$\text{bulk modulus} \quad B := \frac{p}{-dV/V} = \frac{p}{3c} = \frac{\alpha}{3}.$$

Next, consider a shear, in which the displacement might be $\vec{\eta} = cy\hat{e}_x$ produced by forces $\pm F_x$ on the horizontal faces shown, and $\pm F_y$ on the vertical faces. To have no rotation we need $wF_x = LF_y$. The shear modulus G is defined by $-\mathbf{P}_{xy} = F_x/A = Gd\eta_x/dy = Gc$, where A is the area of the top face.



As

$$\mathbf{S}_{xy} = \frac{1}{2}\left(\frac{\partial\eta_x}{\partial y} + \frac{\partial\eta_y}{\partial x}\right) = \frac{1}{2}(c+0) = \frac{c}{2}$$

[6]Representations of a symmetry group are defined as vector spaces which are invariant under the action of the symmetry, and irreducible ones are those for which no proper subspace is closed in that fashion. For more on this, see any book on group theory for physicists. But for representations of the rotation group a course in quantum mechanics may be better. The traceless part of the symmetric tensor transforms like a state with angular momentum 2.

and all other components are zero, we can set

$$\beta = -\frac{\mathbf{P}_{xy}}{\mathbf{S}_{xy}} = 2G.$$

Finally, consider a rod being pulled by a force F stretching a distance ΔL over a length L. **Hooke's constant** is $k = F/\Delta L$ and **Young's modulus** Y is defined by

$$\frac{F}{A} = Y\,\frac{\Delta L}{L} \qquad \text{so } Y = kL/A.$$

The strain $\mathbf{S}_{11} = \Delta L\,/L$, and the stress has $-\mathbf{P}_{11} = Y\mathbf{S}_{11}$, with all other components of the stress zero. But there may be displacement in the transverse directions. If the rod is axially symmetric we may assume $\mathbf{S}_{22} = \mathbf{S}_{33}$, so

$$\begin{aligned} -\operatorname{Tr}\mathbf{P} = -\mathbf{P}_{11} = Y\mathbf{S}_{11} = \alpha\operatorname{Tr}\mathbf{S} = \alpha\left(\mathbf{S}_{11} + 2\mathbf{S}_{22}\right), \\ -\left(\mathbf{P}_{22} - \frac{1}{3}\operatorname{Tr}\mathbf{P}\right) = 0 - \frac{Y}{3}\mathbf{S}_{11} = \beta\left(\mathbf{S}_{22} - \frac{1}{3}\operatorname{Tr}\mathbf{S}\right) = \frac{\beta}{3}\left(\mathbf{S}_{22} - \mathbf{S}_{11}\right) \end{aligned}$$

Thus solving the two equations

$$\begin{aligned} Y\mathbf{S}_{11} &= \alpha\left(\mathbf{S}_{11} + 2\mathbf{S}_{22}\right) \\ -\frac{Y}{3}\mathbf{S}_{11} &= \frac{\beta}{3}\left(\mathbf{S}_{22} - \mathbf{S}_{11}\right) \end{aligned}$$

gives the value of Young's modulus

$$Y = \frac{3\alpha\beta}{2\alpha + \beta}$$

and the contraction of the transverse dimensions,

$$\mathbf{S}_{22} = \frac{\beta - \alpha}{2\alpha + \beta}\,\mathbf{S}_{11}.$$

The Equation of Motion

Now that the generalized Hooke's law provides the forces for a solid in a given configuration, we can write down the equations of motion. The infinitesimal volume originally at the reference point $\vec{r}$ is at position $\vec{r} + \vec{\eta}(\vec{r}, t)$. Its mass is $\int_V \rho dV$, and the force on it is the sum of the volume force and the surface

force. We will write the volume force as $\vec{F}_{\text{vol}} = \int_V \vec{E}(\vec{r})dV$, where E could be $-\rho g \hat{e}_z$ for gravity or some other intensive external force. The surface force is

$$F_i^{\text{surf}} = -\int_S \sum_j \mathbf{P}_{ij}(\vec{r})dS_j \quad \text{or} \quad \vec{F}^{\text{surf}} = -\int_S \mathbf{P}(\vec{r}) \cdot d\vec{S}.$$

In this vector form we imply that the first index of $\mathbf{P}$ is matched to that of $\vec{F}^{\text{surf}}$, while the second index is paired with that of $d\vec{S}$ and summed over. Gauss's law tells us that this is the integral over the volume V of the divergence, but we should take care that this divergence dots the derivative with the second index, that is

$$F_i^{\text{surf}} = -\int_V \sum_j \frac{\partial}{\partial x_j} \mathbf{P}_{ij}(\vec{r})dV.$$

However, as $\mathbf{P}$ is symmetric, we can get away with writing

$$\vec{F}^{\text{surf}} = -\int_V \vec{\nabla} \cdot \mathbf{P}(\vec{r})dV.$$

Writing Hooke's law as

$$\mathbf{P} = -\beta \mathbf{S} + \frac{1}{3}\mathbb{1}(\operatorname{Tr}\mathbf{P} + \beta \operatorname{Tr}\mathbf{S}) = -\beta \mathbf{S} - \frac{1}{3}\mathbb{1}(\alpha - \beta)\operatorname{Tr}\mathbf{S},$$

(where $\mathbb{1}$ is the identity matrix $\mathbb{1}_{ij} = \delta_{ij}$), Newton's second law gives

$$\begin{aligned} \rho(\vec{r})\frac{\partial^2 \vec{\eta}(\vec{r})}{\partial t^2} &= \vec{E}(\vec{r}) - \vec{\nabla} \cdot \mathbf{P}(\vec{r}) \\ &= \vec{E}(\vec{r}) + \beta \vec{\nabla} \cdot \mathbf{S}(\vec{r}) + \frac{\alpha - \beta}{3}\vec{\nabla} \operatorname{Tr}\mathbf{S}(\vec{r}) \end{aligned}$$

where in the last term we note that the divergence contracted into the $\mathbb{1}$ gives an ordinary gradient on the scalar function $\operatorname{Tr}\mathbf{S}$. As the strain tensor is already given in terms of derivatives of $\vec{\eta}$, we have

$$[\vec{\nabla} \cdot \mathbf{S}(\vec{r})]_j = \sum_i \frac{\partial}{\partial x_i}\frac{1}{2}\left(\frac{\partial \eta_i}{\partial x_j} + \frac{\partial \eta_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\vec{\nabla} \cdot \vec{\eta} + \nabla^2 \eta_j\right),$$

or $\vec{\nabla} \cdot \mathbf{S}(\vec{r}) = \frac{1}{2}\vec{\nabla}(\vec{\nabla} \cdot \vec{\eta}) + \frac{1}{2}\nabla^2 \vec{\eta}$. Also $\operatorname{Tr}\mathbf{S} = \sum_i \partial \eta_i / \partial x_i = \vec{\nabla} \cdot \vec{\eta}$, so we find the equations of motion

$$\rho(\vec{r})\frac{\partial^2 \vec{\eta}(\vec{r})}{\partial t^2} = \vec{E}(\vec{r}) + \left(\frac{\alpha}{3} + \frac{\beta}{6}\right)\vec{\nabla}(\vec{\nabla} \cdot \vec{\eta}) + \frac{\beta}{2}\nabla^2 \vec{\eta}. \tag{5.9}$$

This equation is called the Navier equation. We can rewrite this in terms of the shear modulus G and the bulk modulus B:

$$\rho(\vec{r})\frac{\partial^2\vec{\eta}(\vec{r})}{\partial t^2} = \vec{E}(\vec{r}) + \left(B + \frac{G}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{\eta}) + G\nabla^2\vec{\eta}.$$

Fluids

In discussing the motion of pieces of a solid, we specified which piece of the material was under consideration by its "original" or "reference" position $\vec{r}$, from which it might be displaced by a small amount $\vec{\eta}(\vec{r})$. So $\vec{r}$ is actually a label for a particular hunk of material. This is called the **material description**. It is not very useful for a fluid, however, as any element of the fluid may flow arbitrarily far from some initial position. It is more appropriate to consider $\vec{r}$ as a particular point of space, and $\rho(\vec{r},t)$ or $\vec{v}(\vec{r},t)$ or $T(\vec{r},t)$ as the density or velocity or temperature of whatever material happens to be at point $\vec{r}$ at the time t. This is called the **spatial description**.

If we wish to examine how some physical property of the material is changing with time, however, the physical processes which cause change do so on a particular hunk of material. For example, the concentration of a radioactive substance in a hunk of fluid might change due to its decay rate or due to its diffusion, understandable physical processes, while the concentration at the point $\vec{r}$ may change just because new fluid is at the point in question. In describing the physical processes, we will need to consider the rate of change for a given hunk of fluid. Thus we need the stream derivative, which involves the difference of the property (say c) at the new position $\vec{r}\,' = \vec{r} + \vec{v}\Delta t$ at time $t + \Delta t$ and that at the old $\vec{r}, t$. Thus

$$\frac{dc}{dt}(\vec{r},t) = \lim_{\Delta t\to 0}\frac{c(\vec{r}+\vec{v}\Delta t, t+\Delta t) - c(\vec{r},t)}{\Delta t} = \vec{v}\cdot\vec{\nabla}c + \frac{\partial c}{\partial t}.$$

In particular, Newton's law refers to the acceleration of a hunk of material, so it is the stream derivative of the velocity which will be changed by the forces acting on the fluid:

$$\rho(\vec{r})\Delta V\frac{d\vec{v}}{dt} = \rho(\vec{r})\Delta V\left(\vec{v}\cdot\vec{\nabla}\vec{v}(\vec{r},t) + \frac{\partial\vec{v}(\vec{r},t)}{\partial t}\right) = \vec{F}^{\text{surf}} + \vec{F}^{\text{vol}}.$$

The forces on a fluid are different from that in a solid. The volume force is of the same nature, the most common being $\vec{F}^{\text{vol}} = -\rho g\hat{e}_z\,dV$, and the

pressure piece of the stress, $\mathbf{P}_p = +p\mathbb{1}$ is also the same. Thus we can expect part of the force of the form $\vec{F} = (-\rho g\hat{e}_z - \vec{\nabla}\cdot\mathbb{1}p)dV = dV(-\rho g\hat{e}_z - \vec{\nabla}p)$. A static fluid can not experience a shear force. So there will be no shear component of the stress due to a deviatoric part of the strain. But there can be stress due to the velocity of the fluid. Of course a uniformly moving fluid will not be stressed, but if the velocity varies from point to point, stress could be produced. Considering first derivatives, the nine components of $\partial v_i/\partial x_j$ have a scalar piece $\vec{\nabla}\cdot\vec{v}$, an antisymmetric piece, and a traceless symmetric piece, each transforming differently under rotations. Thus for an isotropic fluid the stress may have a piece

$$\mathbf{P}_{ij} = -\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \nu\vec{\nabla}\cdot\vec{v}\mathbb{1}$$

in addition to the scalar, pressure, piece $p\mathbb{1}$. The coefficient μ is called the **viscosity**. The piece proportional to $\vec{\nabla}\cdot\vec{v}$ may be hard to see relative to the pressure term, and is not usually included[7]

The scalar component of $\partial v_i/\partial x_j$, $\vec{\nabla}\cdot\vec{v}$, is in fact just the fractional rate of change of the volume. To see that, consider the surface S which bounds the material in question. If a small piece of that surface is moving with velocity $\vec{v}$, it is adding volume to the material at a rate $\vec{v}\cdot d\vec{S}$, so

$$\frac{dV}{dt} = \oint_S \vec{v}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{v}\, dV.$$

As the mass of the material in question is constant, $d(\rho V)/dt = 0$, so

$$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{v} = 0.$$

This is known as the **equation of continuity**.

With

$$\mathbf{P}_{ij} = p\mathbb{1} - \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \nu\mathbb{1}\sum_j \frac{\partial v_j}{\partial x_j}$$

the surface force is

$$\vec{F}_i^{\text{surf}} = \oint_S -pdS_i + \mu\sum_j \oint_S \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dS_j + \nu\sum_j \oint_S \frac{\partial v_j}{\partial x_j}\, dS_i$$

[7] Tietjens ([19]), following Stokes, assumes the trace of $\mathbf{P}$ is independent of the "velocity of dilatation" $\vec{\nabla}\cdot\vec{v}$, which requires $\nu = -2\mu/3$. But Prandtl and Tietjens [12] drop the $\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$ term in (5.10) entirely, equivalent to taking $\nu = -\mu$.

$$= \int_V \left(-\frac{\partial p}{\partial x_i} + \mu \sum_j \frac{\partial^2 v_i}{\partial x_j^2} + (\mu+\nu) \sum_j \frac{\partial^2 v_j}{\partial x_i x_j} \right) dV$$

where the last equality is by Gauss' law. This can be rewritten in vector form:

$$\vec{F}^{\text{surf}} = \int_V \left(-\vec{\nabla} p + \mu \nabla^2 \vec{v} + (\mu+\nu) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \right) dV$$

Adding in $\vec{F}^{\text{vol}} = -\rho g \hat{e}_z \, dV$ and setting this equal to $\rho \, dV \, d\vec{v}/dt$, we find

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \frac{\partial \vec{v}(\vec{r},t)}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}(\vec{r},t) \qquad (5.10) \\ &= -g\hat{e}_z - \frac{1}{\rho} \vec{\nabla} p(\vec{r},t) + \frac{\mu}{\rho} \nabla^2 \vec{v}(\vec{r},t) + \frac{\mu+\nu}{\rho} \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v}(\vec{r},t) \right). \end{aligned}$$

This is the Navier-Stokes equation for a viscous fluid. For an inviscid fluid, one with a negligible viscosity ($\mu = \nu = 0$), this reduces to the simpler Euler's equation

$$\frac{\partial \vec{v}(\vec{r},t)}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}(\vec{r},t) = -g\hat{e}_z - \frac{1}{\rho} \vec{\nabla} p(\vec{r},t). \qquad (5.11)$$

If we assume the fluid is inviscid and incompressible, so ρ is constant, and also make the further simplifying assumption that we are looking at a steady-state flow, for which $\vec{v}$ and p at a fixed point do not change, the partial derivatives $\partial/\partial t$ vanish, and $\vec{\nabla} \cdot \vec{v} = 0$. Then Euler's equation becomes

$$\vec{v} \cdot \vec{\nabla} \vec{v}(\vec{r}) = -g\hat{e}_z - \frac{1}{\rho} \vec{\nabla} p(\vec{r}). \qquad (5.12)$$

In a steady state situation, any function $f(\vec{r})$ has a stream derivative

$$\frac{d}{dt} f = \vec{v} \cdot \nabla f,$$

so the first term in (5.12) is $d\vec{v}/dt$, and the second term is $-\vec{\nabla}(gz)$. Dotting the equation in this form into $\rho\vec{v}$, we have

$$\rho \vec{v} \cdot \frac{d\vec{v}}{dt} + \rho \vec{v} \cdot \nabla(gz) + \vec{v} \cdot \vec{\nabla} p = 0 = \frac{d}{dt} \left(\frac{1}{2} \rho v^2 + \rho g z + p \right)$$

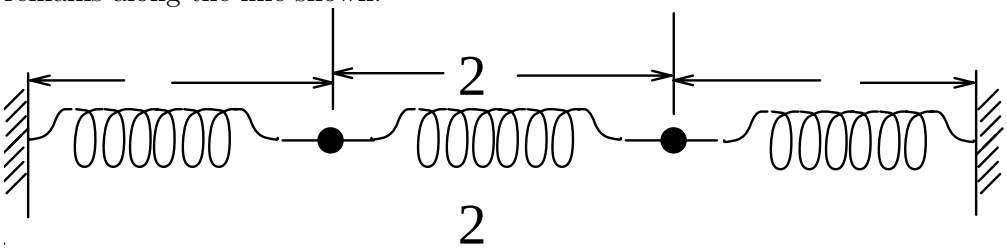
which implies **Bernoulli's equation**:

$$\frac{1}{2} \rho v^2 + \rho g z + p = \text{ constant along a streamline}$$

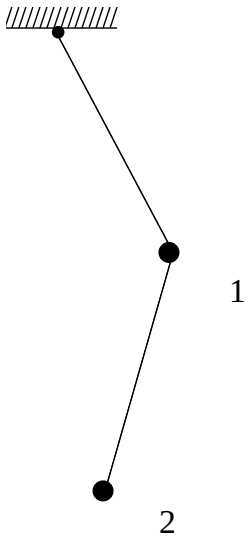
where the restriction is because a streamline is the set of points in the flow which are traversed by an element of the fluid as time goes by.

Exercises

5.1 Three springs connect two masses to each other and to immobile walls, as shown. Find the normal modes and frequencies of oscillation, assuming the system remains along the line shown.



5.2 Consider the motion, in a fixed vertical plane, of a double pendulum consisting of two masses attached to each other and to a fixed point by inextensible strings of length L. The upper mass has mass m_1 and the lower mass m_2. This is all in a laboratory with the ordinary gravitational forces near the surface of the Earth.



a) Set up the Lagrangian for the motion, assuming the strings stay taut.
b) Simplify the system under the approximation that the motion involves only small deviations from equilibrium. Put the problem in matrix form appropriate for the procedure discussed in (5.1).
c) Find the frequencies of the normal modes of oscillation. [Hint: following exactly the steps given in class will be complex, but the analogous procedure reversing the order of U and T will work easily.]

5.3 (a) Show that if three mutually gravitating point masses are at the vertices of an equilateral triangle which is rotating about an axis normal to the plane of the triangle and through the center of mass, at a suitable angular velocity ω, this motion satisfies the equations of motion. Thus this configuration is an equilibrium in the rotating coordinate system. Do not assume the masses are equal.
(b) Suppose that two stars of masses M_1 and M_2 are rotating in circular orbits about their common center of mass. Consider a small mass m which is approximately in the equilibrium position described above (which is known as the L_5 point). The mass is small enough that you can ignore its effect on the two stars.

Analyze the motion, considering specifically the stability of the equilibrium point as a function of the ratio of the masses of the stars.

5.4 In considering the limit of a loaded string we found that in the limit $a \to 0, n \to \infty$ with ℓ fixed, the modes with fixed integer p became a smooth excitation $y(x,t)$ with finite wavenumber k and frequency $\omega = ck$.
Now consider the limit with $q := n+1-p$ fixed as $n \to \infty$. Calculate the expression for y_j in that limit. This will not have a smooth limit, but there is nonetheless a sense in which it can be described by a finite wavelength. Explain what this is, and give the expression for y_j in terms of this wavelength.

5.5 Consider the Navier equation ignoring the volume force, and show that
a) a uniform elastic material can support longitudinal waves. At what speed do they travel?
b) an uniform elastic material can support transverse waves. At what speed do they travel?
c) Granite has a density of 2700 kg/m^3, a bulk modulus of 4×10^{10}N/m^2 and a shear modulus of 2.5×10^{10}N/m^2. If a short spike of transverse oscillations arrives 25 seconds after a similar burst of longitudinal oscillations, how far away was the explosion that caused these waves?