- multipole moments of the sun

- forces due to other planets

- effects of corrections to Newtonian gravity due to general relativity

- friction due to the solar wind and gas in the solar system

Learning how to estimate or incorporate such effects is not trivial.

Secondly, classical mechanics is not a dead field of research — in fact, in the last few decades there has been a great deal of interest in "dynamical systems". Attention has shifted from calculation of the trajectory over fixed intervals of time to questions of the long-term stability of the motion. New ways of looking at dynamical behavior have emerged, such as chaos and fractal systems.

Thirdly, the fundamental concepts of classical mechanics provide the conceptual framework of quantum mechanics. For example, although the Hamiltonian and Lagrangian were developed as sophisticated techniques for performing classical mechanics calculations, they provide the basic dynamical objects of quantum mechanics and quantum field theory respectively. One view of classical mechanics is as a steepest path approximation to the path integral which describes quantum mechanics. This integral over paths is of a classical quantity depending on the "action" of the motion.

So classical mechanics is worth learning well, and we might as well jump right in.

1.2 Single Particle Kinematics

We start with the simplest kind of system, a single unconstrained particle, free to move in three dimensional space, under the influence of a force $\vec{F}$.

1.2.1 Motion in configuration space

The motion of the particle is described by a function which gives its position as a function of time. These positions are points in Euclidean space. Euclidean space is similar to a vector space, except that there is no special point which is fixed as the origin. It does have a metric, that is, a notion of distance between any two points, $D(A, B)$. It also has the concept of a displacement $A - B$ from one point B in the Euclidean space to another,

A. These displacements do form a vector space, and for a three-dimensional Euclidean space, the vectors form a three-dimensional real vector space $\mathbb{R}^3$, which can be given an orthonormal basis such that the distance between A and B is given by $D(A, B) = \sqrt{\sum_{i=1}^{3}[(A - B)_i]^2}$. Because the mathematics of vector spaces is so useful, we often convert our Euclidean space to a vector space by choosing a particular point as the origin. Each particle's position is then equated to the displacement of that position from the origin, so that it is described by a position vector $\vec{r}$ relative to this origin. But the origin has no physical significance unless it has been choosen in some physically meaningful way. In general the multiplication of a position vector by a scalar is as meaningless physically as saying that 42nd street is three times 14th street. The cartesian components of the vector $\vec{r}$, with respect to some fixed though arbitrary coordinate system, are called the coordinates, cartesian coordinates in this case. We shall find that we often (even usually) prefer to change to other sets of coordinates, such as polar or spherical coordinates, but for the time being we stick to cartesian coordinates.

The motion of the particle is the function $\vec{r}(t)$ of time. Certainly one of the central questions of classical mechanics is to determine, given the physical properties of a system and some initial conditions, what the subsequent motion is. The required "physical properties" is a specification of the force, $\vec{F}$. The beginnings of modern classical mechanics was the realization early in the 17th century that the physics, or dynamics, enters into the motion (or kinematics) through the force and its effect on the acceleration, and not through any direct effect of dynamics on the position or velocity of the particle.

Most likely the force will depend on the position of the particle, say for a particle in the gravitational field of a fixed (heavy) source at the origin, for which

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^3}\vec{r}. \tag{1.1}$$

But the force might also depend explicitly on time. For example, for the motion of a spaceship near the Earth, we might assume that the force is given by sum of the Newtonian gravitational forces of the Sun, Moon and Earth. Each of these forces depends on the positions of the corresponding heavenly body, which varies with time. The assumption here is that the motion of these bodies is independent of the position of the light spaceship. We assume someone else has already performed the nontrivial problem of finding the positions of these bodies as functions of time. Given that, we

can write down the force the spaceship feels at time t if it happens to be at position $\vec{r}$,

$$\vec{F}(\vec{r}, t) = -GmM_S \frac{\vec{r} - \vec{R}_S(t)}{|r - R_S(t)|^3} - GmM_E \frac{\vec{r} - \vec{R}_E(t)}{|r - R_E(t)|^3}$$
$$- GmM_M \frac{\vec{r} - \vec{R}_M(t)}{|r - R_M(t)|^3}.$$

So there is an explicit dependence on t Finally, the force might depend on the velocity of the particle, as for example for the Lorentz force on a charged particle in electric and magnetic fields

$$\vec{F}(\vec{r}, \vec{v}, t) = q\,\vec{E}(\vec{r}, t) + q\,\vec{v} \times \vec{B}(\vec{r}, t). \tag{1.2}$$

However the force is determined, it determines the motion of the particle through the second order differential equation known as Newton's Second Law

$$\vec{F}(\vec{r}, \vec{v}, t) = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}.$$

As this is a second order differential equation, the solution depends in general on two arbitrary (3-vector) parameters, which we might choose to be the initial position and velocity, $\vec{r}(0)$ and $\vec{v}(0)$.

For a given physical situation and a given set of initial conditions for the particle, Newton's laws determine the motion $\vec{r}(t)$, which is a curve in **configuration space** parameterized by time t, known as the **trajectory** in configuration space. If we consider the curve itself, independent of how it depends on time, this is called the **orbit** of the particle. For example, the orbit of a planet, in the approximation that it feels only the field of a fixed sun, is an ellipse. That word does not imply any information about the time dependence or parameterization of the curve.

1.2.2 Conserved Quantities

While we tend to think of Newtonian mechanics as centered on Newton's Second Law in the form $\vec{F} = m\vec{a}$, he actually started with the observation that in the absence of a force, there was uniform motion. We would now say that under these circumstances the **momentum** $\vec{p}(t)$ is **conserved**, $d\vec{p}/dt =$

0. In his second law, Newton stated the effect of a force as producing a rate of change of momentum, which we would write as

$$\vec{F} = d\vec{p}/dt,$$

rather than as producing an acceleration $\vec{F} = m\vec{a}$. In focusing on the concept of momentum, Newton emphasized one of the fundamental quantities of physics, useful beyond Newtonian mechanics, in both relativity and quantum mechanics[1]. Only after using the classical relation of momentum to velocity, $\vec{p} = m\vec{v}$, and the assumption that m is constant, do we find the familiar $\vec{F} = m\vec{a}$.

One of the principal tools in understanding the motion of many systems is isolating those quantities which do not change with time. A **conserved quantity** is a function of the positions and momenta, and perhaps explicitly of time as well, $Q(\vec{r}, \vec{p}, t)$, which remains unchanged when evaluated along the actual motion, $dQ(\vec{r}(t), \vec{p}(t), t)/dt = 0$. A function depending on the positions, momenta, and time is said to be a function on *extended phase space*[2]. When time is not included, the space is called *phase space*. In this language, a conserved quantity is a function on extended phase space with a vanishing total time derivative along any path which describes the motion of the system.

A single particle with no forces acting on it provides a very simple example. As Newton tells us, $\dot{\vec{p}} = d\vec{p}/dt = \vec{F} = 0$, so the momentum is conserved. There are three more conserved quantities $\vec{Q}(\vec{r}, \vec{p}, t) := \vec{r}(t) - t\vec{p}(t)/m$, which have a time rate of change $d\vec{Q}/dt = \dot{\vec{r}} - \vec{p}/m - t\dot{\vec{p}}/m = 0$. These six independent conserved quantities are as many as one could have for a system with a six dimensional phase space, and they completely solve for the motion. Of course this was a very simple system to solve. We now consider a particle under the influence of a force.

Energy

Consider a particle under the influence of an external force $\vec{F}$. In general, the momentum will not be conserved, although if any cartesian component of the force vanishes along the motion, that component of the momentum

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[1]The relationship of momentum to velocity is changed in these extensions, however.
[2]Phase space is discussed further in section 1.4.

will be conserved. Also the **kinetic energy**, defined as $T = \frac{1}{2}m\vec{v}^2$, will not in general be conserved, because

$$\frac{dT}{dt} = m\dot{\vec{v}} \cdot \vec{v} = \vec{F} \cdot \vec{v}.$$

As the particle moves from the point $\vec{r}_i$ to the point $\vec{r}_f$ the total change in the kinetic energy is the **work** done by the force $\vec{F}$,

$$\Delta T = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}.$$

If the force law $\vec{F}(\vec{r}, \vec{p}, t)$ applicable to the particle is independent of time and velocity, then the work done will not depend on how quickly the particle moved along the path from $\vec{r}_i$ to $\vec{r}_f$. If in addition the work done is independent of the path taken between these points, so it depends only on the endpoints, then the force is called a **conservative force** and we associate with it **potential energy**

$$U(\vec{r}) = U(\vec{r}_0) + \int_{\vec{r}}^{\vec{r}_0} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,',$$

where $\vec{r}_0$ is some arbitrary reference position and $U(\vec{r}_0)$ is an arbitrarily chosen reference energy, which has no physical significance in ordinary mechanics. $U(\vec{r})$ represents the potential the force has for doing work on the particle if the particle is at position $\vec{r}$.

The condition for the path integral to be independent of the path is that it gives the same results along any two coterminous paths Γ_1 and Γ_2, or alternatively that it give zero when evaluated along any closed path such as $\Gamma = \Gamma_1 - \Gamma_2$, the path consisting of following Γ_1 and then taking Γ_2 backwards to the starting point. By Stokes' Theorem, this line integral is equivalent to an integral over any surface S bounded by Γ,

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \int_S \vec{\nabla} \times \vec{F} \, dS.$$
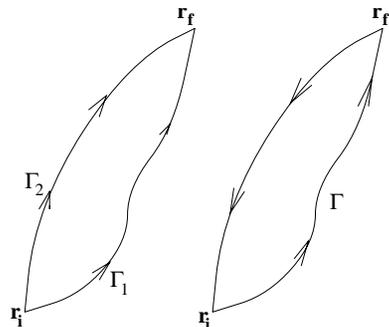
Independence of path $\int_{\Gamma_1} = \int_{\Gamma_2}$ is equivalent to vanishing of the path integral over closed paths Γ, which is in turn equivalent to the vanishing of the curl on the surface whose boundary is Γ.

Thus the requirement that the integral of $\vec{F} \cdot d\vec{r}$ vanish around *any* closed path is equivalent to the requirement that the curl of $\vec{F}$ vanish everywhere in space.

By considering an infinitesimal path from $\vec{r}$ to $\vec{r} + \Delta\vec{r}$, we see that

$$U(\vec{r} + \vec{\Delta}) - U(\vec{r}) = -\vec{F} \cdot \Delta\vec{r}, \quad \text{or}$$
$$\vec{F}(r) = -\vec{\nabla}U(r).$$

The value of the concept of potential energy is that it enables finding a conserved quantity, the total energy, in situtations in which all forces are conservative. Then the total energy $E = T + U$ changes at a rate

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{\nabla}U = \vec{F} \cdot \vec{v} - \vec{v} \cdot \vec{F} = 0.$$

The total energy can also be used in systems with both conservative and non-conservative forces, giving a quantity whose rate of change is determined by the work done only by the nonconservative forces. One example of this usefulness is in the discussion of a slightly damped harmonic oscillator driven by a periodic force near resonance. Then the amplitude of steady-state motion is determined by a balence between the average power input by the driving force and the average power dissipated by friction, the two nonconservative forces in the problem, without needing to worry about the work done by the spring.

Angular momentum

Another quantity which is often useful because it may be conserved is the angular momentum. The definition requires a reference point in the Euclidean space, say $\vec{r}_0$. Then a particle at position $\vec{r}$ with momentum $\vec{p}$ has an **angular momentum** about $\vec{r}_0$ given by $\vec{L} = (\vec{r} - \vec{r}_0) \times \vec{p}$. Very often we take the reference point $\vec{r}_0$ to be the same as the point we have chosen as the origin in converting the Euclidian space to a vector space, so $\vec{r}_0 = 0$, and

$$\vec{L} = \vec{r} \times \vec{p}$$
$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \frac{1}{m}\vec{p} \times \vec{p} + \vec{r} \times \vec{F} = 0 + \vec{\tau} = \vec{\tau}.$$

where we have defined the **torque** about $\vec{r}_0$ as $\tau = (\vec{r} - \vec{r}_0) \times \vec{F}$ in general, and $\tau = \vec{r} \times \vec{F}$ when our reference point $\vec{r}_0$ is at the origin.

We see that if the torque $\vec{\tau}(t)$ vanishes (at all times) the angular momentum is conserved. This can happen not only if the force is zero, but also if the force always points to the reference point. This is the case in a central force problem such as motion of a planet about the sun.

1.3 Systems of Particles

So far we have talked about a system consisting of only a single particle, possibly influenced by external forces. Consider now a system of n particles with positions $\vec{r}_i$, $i = 1, \ldots, n$, in flat space. The configuration of the system then has $3n$ coordinates (configuration space is $\mathbb{R}^{3n}$), and the phase space has $6n$ coordinates $\{\vec{r}_i, \vec{p}_i\}$.

1.3.1 External and internal forces

Let $\vec{F}_i$ be the total force acting on particle i. It is the sum of the forces produced by each of the other particles and that due to any external force. Let $\vec{F}_{ji}$ be the force particle j exerts on particle i and let $\vec{F}_i^E$ be the external force on particle i. Using Newton's second law on particle i, we have

$$\vec{F}_i = \vec{F}_i^E + \sum_j \vec{F}_{ji} = \dot{\vec{p}}_i = m_i \dot{\vec{v}}_i,$$

where m_i is the mass of the i'th particle. Here we are assuming forces have identifiable causes, which is the real meaning of Newton's second law, and that the causes are either individual particles or external forces. Thus we are assuming there are no "three-body" forces which are not simply the sum of "two-body" forces that one object exerts on another.

Define the **center of mass** and **total mass**

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i}, \qquad M = \sum m_i.$$

Then if we define the **total momentum**

$$\vec{P} = \sum \vec{p}_i = \sum m_i \vec{v}_i = \frac{d}{dt} \sum m_i \vec{r}_i = M \frac{d\vec{R}}{dt},$$

we have

$$\frac{d\vec{P}}{dt} = \dot{\vec{P}} = \sum \dot{\vec{p}}_i = \sum \vec{F}_i = \sum_i \vec{F}_i^E + \sum_{ij} \vec{F}_{ji}.$$

Let us define $\vec{F}^E = \sum_i \vec{F}_i^E$ to be the **total external force**. If Newton's Third Law holds,

$$\vec{F}_{ji} = -\vec{F}_{ij}, \text{ so } \sum_{ij} \vec{F}_{ij} = 0, \text{ and}$$

$$\dot{\vec{P}} = \vec{F}^E. \tag{1.3}$$

Thus the internal forces cancel in pairs in their effect on the total momentum, which changes only in response to the total external force. As an obvious but very important consequence[3] *the total momentum of an isolated system is conserved.*

The total angular momentum is also just a sum over the individual angular momenta, so for a system of point particles,

$$\vec{L} = \sum \vec{L}_i = \sum \vec{r}_i \times \vec{p}_i.$$

Its rate of change with time is

$$\frac{d\vec{L}}{dt} = \dot{\vec{L}} = \sum_i \vec{v}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \vec{F}_i = 0 + \sum_i \vec{r}_i \times \vec{F}_i^E + \sum_{ij} \vec{r}_i \times \vec{F}_{ji}.$$

---

[3]There are situations and ways of describing them in which the law of action and reaction seems not to hold. For example, a current i_1 flowing through a wire segment $d\vec{s}_1$ contributes, according to the law of Biot and Savart, a magnetic field $d\vec{B} = \mu_0 i_1 d\vec{s}_1 \times \vec{r}/4\pi |r|^3$ at a point $\vec{r}$ away from the current element. If a current i_2 flows through a segment of wire $d\vec{s}_2$ at that point, it feels a force

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} i_1 i_2 \frac{d\vec{s}_2 \times (d\vec{s}_1 \times \vec{r})}{|r|^3}$$

due to element 1. On the other hand $\vec{F}_{21}$ is given by the same expression with $d\vec{s}_1$ and $d\vec{s}_2$ interchanged and the sign of $\vec{r}$ reversed, so

$$\vec{F}_{12} + \vec{F}_{21} = \frac{\mu_0}{4\pi} \frac{i_1 i_2}{|r|^3} \left[d\vec{s}_1 (d\vec{s}_2 \cdot \vec{r}) - d\vec{s}_2 (d\vec{s}_1 \cdot \vec{r}) \right],$$

which is not generally zero. One should not despair for the validity of momentum conservation. The Law of Biot and Savart only holds for time-independent current distributions. Unless the currents form closed loops, there will be a charge buildup and Coulomb forces need to be considered. If the loops are closed, the total momentum will involve integrals over the two closed loops, for which $\int\int F_{12} + F_{21}$ can be shown to vanish. More generally, even the sum of the momenta of the current elements is not the whole story, because there is momentum in the electromagnetic field, which will be changing in the time-dependent situation.

The total external torque is naturally defined as

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i^E,$$

so we might ask if the last term vanishes due the Third Law, which permits us to rewrite $\vec{F}_{ji} = \frac{1}{2}\left(\vec{F}_{ji} - \vec{F}_{ij}\right)$. Then the last term becomes

$$
\begin{aligned}
\sum_{ij} \vec{r}_i \times \vec{F}_{ji} &= \frac{1}{2}\sum_{ij} \vec{r}_i \times \vec{F}_{ji} - \frac{1}{2}\sum_{ij} \vec{r}_i \times \vec{F}_{ij} \\
&= \frac{1}{2}\sum_{ij} \vec{r}_i \times \vec{F}_{ji} - \frac{1}{2}\sum_{ij} \vec{r}_j \times \vec{F}_{ji} \\
&= \frac{1}{2}\sum_{ij} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}.
\end{aligned}
$$

This is not automatically zero, but vanishes if one assumes a stronger form of the Third Law, namely that the action and reaction forces between two particles acts along the line of separation of the particles. If the force law is independent of velocity and rotationally and translationally symmetric, there is no other direction for it to point. For spinning particles and magnetic forces the argument is not so simple — in fact electromagnetic forces between moving charged particles are really only correctly viewed in a context in which the system includes not only the particles but also the fields themselves. For such a system, in general the total energy, momentum, and angular momentum of the particles alone will not be conserved, because the fields can carry all of these quantities. But properly defining the energy, momentum, and angular momentum of the electromagnetic fields, and including them in the totals, will result in quantities conserved as a result of symmetries of the underlying physics. This is further discussed in section 8.3.

Making the assumption that the strong form of Newton's Third Law holds, we have shown that

$$\vec{\tau} = \frac{d\vec{L}}{dt}. \tag{1.4}$$

The conservation laws are very useful because they permit algebraic solution for part of the velocity. Taking a single particle as an example, if $E = \frac{1}{2}mv^2 + U(\vec{r})$ is conserved, the speed $|v(t)|$ is determined at all times (as a function of $\vec{r}$) by one arbitrary constant E. Similarly if $\vec{L}$ is conserved,

the components of $\vec{v}$ which are perpendicular to $\vec{r}$ are determined in terms of the fixed constant $\vec{L}$. With both conserved, $\vec{v}$ is completely determined except for the sign of the radial component. Examples of the usefulness of conserved quantities are everywhere, and will be particularly clear when we consider the two body central force problem later. But first we continue our discussion of general systems of particles.

As we mentioned earlier, the total angular momentum depends on the point of evaluation, that is, the origin of the coordinate system used. We now show that it consists of two contributions, the angular momentum about the center of mass and the angular momentum of a fictitious point object located at the center of mass. Let $\vec{r}_i'$ be the position of the i'th particle with respect to the center of mass, so $\vec{r}_i' = \vec{r}_i - \vec{R}$. Then

$$
\begin{aligned}
\vec{L} &= \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \left(\vec{r}_i' + \vec{R}\right) \times \left(\dot{\vec{r}}_i' + \dot{\vec{R}}\right) \\
&= \sum_i m_i \vec{r}_i' \times \dot{\vec{r}}_i' + \sum_i m_i \vec{r}_i' \times \dot{\vec{R}} \\
&\quad + \vec{R} \times \sum_i m_i \dot{\vec{r}}_i' + M\vec{R} \times \dot{\vec{R}} \\
&= \sum_i \vec{r}_i' \times \vec{p}_i' + \vec{R} \times \vec{P}.
\end{aligned}
$$

Here we have noted that $\sum m_i \vec{r}_i' = 0$, and also its derivative $\sum m_i \vec{v}_i' = 0$. We have defined $\vec{p}_i' = m_i \vec{v}_i'$, the momentum in the center of mass reference frame. The first term of the final form is the sum of the angular momenta of the particles *about their center of mass*, while the second term is the angular momentum the system would have if it were collapsed to a point at the center of mass. Notice we did not need to assume the center of mass is unaccelerated.

What about the total energy? The kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i \left(\vec{v}_i' + \vec{V}\right) \cdot \left(\vec{v}_i' + \vec{V}\right) \\
&= \frac{1}{2}\sum m_i v_i'^2 + \frac{1}{2}MV^2, \tag{1.5}
\end{aligned}
$$

where $\vec{V} = \dot{\vec{R}}$ is the velocity of the center of mass. The cross term vanishes once again, because $\sum m_i \vec{v}_i' = 0$. Thus the kinetic energy of the system can also be viewed as the sum of the kinetic energies of the constituents about

the center of mass, plus the kinetic energy the system would have if it were collapsed to a particle at the center of mass.

If the forces on the system are due to potentials, the total energy will be conserved, but this includes not only the potential due to the external forces but also that due to interparticle forces, $\sum U_{ij}(\vec{r}_i, \vec{r}_j)$. In general this contribution will not be zero or even constant with time, and the internal potential energy will need to be considered. One exception to this is the case of a rigid body.

1.3.2 Constraints

A rigid body is defined as a system of n particles for which all the interparticle distances are constrained to fixed constants, $|\vec{r}_i - \vec{r}_j| = c_{ij}$, and the interparticle potentials are functions only of these interparticle distances. As these distances do not vary, neither does the internal potential energy. These interparticle forces cannot do work, and the internal potential energy may be ignored.

The rigid body is an example of a constrained system, in which the general $3n$ degrees of freedom are restricted by some forces of constraint which place conditions on the coordinates $\vec{r}_i$, perhaps in conjunction with their momenta. In such descriptions we do not wish to consider or specify the forces themselves, but only their (approximate) effect. The forces are assumed to be whatever is necessary to have that effect. It is generally assumed, as in the case with the rigid body, that the constraint forces do no work under displacements allowed by the constraints. We will consider this point in more detail later.

If the constraints can be phrased so that they are on the coordinates and time only, as $\Phi_i(\vec{r}_1, ...\vec{r}_n, t) = 0, i = 1, \ldots, k$, they are known as **holonomic constraints**. These constraints determine hypersurfaces in configuration space to which all motion of the system is confined. In general this hypersurface forms a $3n - k$ dimensional manifold. We might describe the configuration point on this manifold in terms of $3n - k$ generalized coordinates, $q_j, j = 1, \ldots, 3n - k$, so that the $3n - k$ variables q_j, together with the k constraint conditions $\Phi_i(\{\vec{r}_i\}) = 0$, determine the $\vec{r}_i = \vec{r}_i(q_1, \ldots, q_{3n-k}, t)$

The constrained subspace of configuration space need not be a flat space. Consider, for example, a mass on one end of a rigid light rod of length L, the other end of which is fixed to be at the origin $\vec{r} = 0$, though the rod is completely free to rotate. Clearly the possible values of the cartesian coordinates $\vec{r}$ of the position of the mass satisfy the constraint $|\vec{r}| = L$, so $\vec{r}$ lies on the surface of a sphere of radius L. We might choose as generalized coordinates the standard spherical angles θ and ϕ. Thus the constrained subspace is two dimensional but not flat — rather it is the surface of a sphere, which mathematicians call S^2. It is natural to reexpress the dynamics in terms of θ and ϕ.
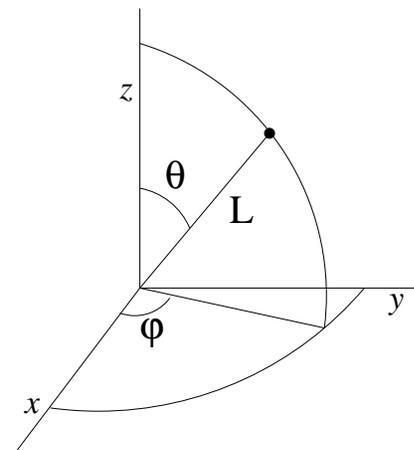
Generalized coordinates (θ, ϕ) for a particle constrained to lie on a sphere.
[Note: mathematics books tend to interchange θ and ϕ from the choice we use here, which is what most physics books use.]

Note that with this constrained configuration space, we see that ideas common in Euclidean space are no longer clear. The displacement between two points A and B, as a three vector, cannot be added to a general point C, and in two dimensions, a change, for example, of $\Delta\phi$ is a very differnent change in configuration depending on what θ is.

The use of generalized (non-cartesian) coordinates is not just for constrained systems. The motion of a particle in a central force field about the origin, with a potential $U(\vec{r}) = U(|\vec{r}|)$, is far more naturally described in terms of spherical coordinates r, θ, and ϕ than in terms of x, y, and z.

Before we pursue a discussion of generalized coordinates, it must be pointed out that not all constraints are holonomic. The standard example is a disk of radius R, which rolls on a fixed horizontal plane. It is constrained to always remain vertical, and also to roll without slipping on the plane. As coordinates we can choose the x and y of the center of the disk, which are also the x and y of the contact point, together with the angle a fixed line on the disk makes with the downward direction, ϕ, and the angle the axis of the disk makes with the x axis, θ.

As the disk rolls through an angle $d\phi$, the point of contact moves a distance $Rd\phi$ in a direction depending on θ,

$$Rd\phi\sin\theta = dx$$
$$Rd\phi\cos\theta = dy$$

Dividing by dt, we get two constraints involving the positions *and* velocities,
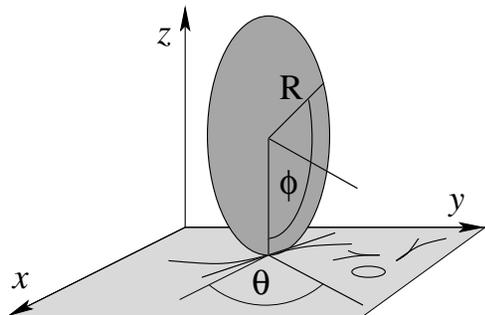
$$\Phi_1 := R\dot\phi\sin\theta - \dot x = 0$$
$$\Phi_2 := R\dot\phi\cos\theta - \dot y = 0.$$

The fact that these involve velocities does not automatically make them nonholonomic. In the simpler one-dimensional problem in which the disk is confined to the yz plane, rolling

A vertical disk free to roll on a plane. A fixed line on the disk makes an angle of ϕ with respect to the vertical, and the axis of the disk makes an angle θ with the x-axis. The long curved path is the trajectory of the contact point. The three small paths are alternate trajectories illustrating that x, y, and ϕ can each be changed without any net change in the other coordinates.

along $x = 0$ ($\theta = 0$), we would have only the coordinates ϕ and y, with the rolling constraint $R\dot\phi - \dot y = 0$. But this constraint can be integrated, $R\phi(t) - y(t) = c$, for some constant c, so that it becomes a constraint among just the coordinates, and is holomorphic. This cannot be done with the two-dimensional problem. We can see that there is no constraint among the four coordinates themselves because each of them can be changed by a motion which leaves the others unchanged. Rotating θ without moving the other coordinates is straightforward. By rolling the disk along each of the three small paths shown to the right of the disk, we can change one of the variables x, y, or ϕ, respectively, with no net change in the other coordinates. Thus all values of the coordinates[4] can be achieved in this fashion.

There are other, less interesting, nonholonomic constraints given by inequalities rather than constraint equations. A bug sliding down a bowling

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[4]Thus the configuration space is $x \in \mathbb{R}$, $y \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$, or, if we allow more carefully for the continuity as θ and ϕ go through 2π, the more accurate statement is that configuration space is $\mathbb{R}^2 \times (S^1)^2$, where S^1 is the circumference of a circle, $\theta \in [0, 2\pi]$, with the requirement that $\theta = 0$ is equivalent to $\theta = 2\pi$.

ball obeys the constraint $|\vec r| \geq R$. Such problems are solved by considering the constraint with an equality ($|\vec r| = R$), but restricting the region of validity of the solution by an inequality on the constraint force ($N \geq 0$), and then supplementing with the unconstrained problem once the bug leaves the surface.

In quantum field theory, anholonomic constraints which are functions of the positions and momenta are further subdivided into first and second class constraints *à la* Dirac, with the first class constraints leading to local gauge invariance, as in Quantum Electrodynamics or Yang-Mills theory. But this is heading far afield.

1.3.3 Generalized Coordinates for Unconstrained Systems

Before we get further into constrained systems and D'Alembert's Principle, we will discuss the formulation of a conservative unconstrained system in generalized coordinates. Thus we wish to use $3n$ generalized coordinates q_j, which, together with time, determine all of the $3n$ cartesian coordinates $\vec r_i$:

$$\vec r_i = \vec r_i(q_1, ..., q_{3n}, t).$$

Notice that this is a relationship between different descriptions of the same point in configuration space, and the functions $\vec r_i(\{q\}, t)$ are independent of the motion of any particle. We are assuming that the $\vec r_i$ and the q_j are each a complete set of coordinates for the space, so the q's are also functions of the $\{\vec r_i\}$ and t:

$$q_j = q_j(\vec r_1, ..., \vec r_n, t).$$

The t dependence permits there to be an explicit dependence of this relation on time, as we would have, for example, in relating a rotating coordinate system to an inertial cartesian one.

Let us change the cartesian coordinate notation slightly, with $\{x_k\}$ the $3n$ cartesian coordinates of the n 3-vectors $\vec r_i$, deemphasizing the division of these coordinates into triplets.

A small change in the coordinates of a particle in configuration space, whether an actual change over a small time interval dt or a "virtual" change between where a particle is and where it might have been under slightly altered circumstances, can be described by a set of δx_k or by a set of δq_j. If

we are talking about a virtual change at the same time, these are related by the chain rule

$$\delta x_k = \sum_j \frac{\partial x_k}{\partial q_j} \delta q_j, \quad \delta q_j = \sum_k \frac{\partial q_j}{\partial x_k} \delta x_k, \quad (\text{for } \delta t = 0). \tag{1.6}$$

For the actual motion through time, or any variation where δt is not assumed to be zero, we need the more general form,

$$\delta x_k = \sum_j \frac{\partial x_k}{\partial q_j} \delta q_j + \frac{\partial x_k}{\partial t} \delta t, \quad \delta q_j = \sum_k \frac{\partial q_j}{\partial x_k} \delta x_k + \frac{\partial q_k}{\partial t} \delta t. \tag{1.7}$$

A virtual displacement, with $\delta t = 0$, is the kind of variation we need to find the forces described by a potential. Thus the force is

$$F_k = -\frac{\partial U(\{x\})}{\partial x_k} = -\sum_j \frac{\partial U(\{x(\{q\})\})}{\partial q_j} \frac{\partial q_j}{\partial x_k} = \sum_j \frac{\partial q_j}{\partial x_k} Q_j, \tag{1.8}$$

where

$$Q_j := \sum_k F_k \frac{\partial x_k}{\partial q_j} = -\frac{\partial U(\{x(\{q\})\})}{\partial q_j} \tag{1.9}$$

is known as the **generalized force**. We may think of $\tilde{U}(q,t) := U(x(q),t)$ as a potential in the generalized coordinates $\{q\}$. Note that if the coordinate transformation is time-dependent, it is possible that a time-independent potential $U(x)$ will lead to a time-dependent potential $\tilde{U}(q,t)$, and a system with forces described by a time-dependent potential is not conservative.

The definition of the generalized force Q_j in the left part of (1.9) holds even if the cartesian force is not described by a potential.

The q_k do not necessarily have units of distance. For example, one q_k might be an angle, as in polar or spherical coordinates. The corresponding component of the generalized force will have the units of energy and we might consider it a torque rather than a force.

1.3.4 Kinetic energy in generalized coordinates

We have seen that, under the right circumstances, the potential energy can be thought of as a function of the generalized coordinates q_k, and the generalized

forces Q_k are given by the potential just as for ordinary cartesian coordinates and their forces. Now we examine the kinetic energy

$$T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2} \sum_j m_j \dot{x}_j^2$$

where the $3n$ values m_j are not really independent, as each particle has the same mass in all three dimensions in ordinary Newtonian mechanics[5]. Now

$$\dot{x}_j = \lim_{\Delta t \to 0} \frac{\Delta x_j}{\Delta t} = \lim_{\Delta t \to 0} \left(\sum_k \left. \frac{\partial x_j}{\partial q_k} \right|_{q,t} \frac{\Delta q_k}{\Delta t} \right) + \left. \frac{\partial x_j}{\partial t} \right|_q,$$

where $|_{q,t}$ means that t and the q's other than q_k are held fixed. The last term is due to the possibility that the coordinates $x_i(q_1, ..., q_{3n}, t)$ may vary with time even for fixed values of q_k. So the chain rule is giving us

$$\dot{x}_j = \frac{dx_j}{dt} = \sum_k \left. \frac{\partial x_j}{\partial q_k} \right|_{q,t} \dot{q}_k + \left. \frac{\partial x_j}{\partial t} \right|_q. \tag{1.10}$$

Plugging this into the kinetic energy, we see that

$$T = \frac{1}{2} \sum_{j,k,\ell} m_j \frac{\partial x_j}{\partial q_k} \frac{\partial x_j}{\partial q_\ell} \dot{q}_k \dot{q}_\ell + \sum_{j,k} m_j \frac{\partial x_j}{\partial q_k} \dot{q}_k \left. \frac{\partial x_j}{\partial t} \right|_q + \frac{1}{2} \sum_j m_j \left(\left. \frac{\partial x_j}{\partial t} \right|_q \right)^2. \tag{1.11}$$

What is the interpretation of these terms? Only the first term arises if the relation between x and q is time independent. The second and third terms are the sources of the $\vec{r} \cdot (\vec{\omega} \times \vec{r})$ and $(\vec{\omega} \times \vec{r})^2$ terms in the kinetic energy when we consider rotating coordinate systems[6].

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[5]But in an anisotropic crystal, the effective mass of a particle might in fact be different in different directions.

[6]This will be fully developed in section 4.2

Let's work a simple example: we will consider a two dimensional system using polar coordinates with θ measured from a direction rotating at angular velocity ω. Thus the angle the radius vector to an arbitrary point (r, θ) makes with the inertial x_1-axis is $\theta + \omega t$, and the relations are

$$\begin{aligned} x_1 &= r\cos(\theta + \omega t), \\ x_2 &= r\sin(\theta + \omega t), \end{aligned}$$

with inverse relations

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2}, \\ \theta &= \sin^{-1}(x_2/r) - \omega t. \end{aligned}$$

So $\dot{x}_1 = \dot{r}\cos(\theta + \omega t) - \dot{\theta}r\sin(\theta + \omega t) - \omega r\sin(\theta + \omega t)$, where the last term is from $\partial x_j/\partial t$, and $\dot{x}_2 = \dot{r}\sin(\theta + \omega t) + \dot{\theta}r\cos(\theta + \omega t) + \omega r\cos(\theta + \omega t)$. In the square, things get a bit simpler, $\sum \dot{x}_i^2 = \dot{r}^2 + r^2(\omega + \dot{\theta})^2$.

We see that the form of the kinetic energy in terms of the generalized coordinates and their velocities is much more complicated than it is in cartesian inertial coordinates, where it is coordinate independent, and a simple diagonal quadratic form in the velocities. In generalized coordinates, it is quadratic but not homogeneous[7] in the velocities, and with an arbitrary dependence on the coordinates. In general, even if the coordinate transformation is time **independent**, the form of the kinetic energy is still coordinate dependent and, while a purely quadratic form in the velocities, it is not necessarily diagonal. In this time-independent situation, we have

$$T = \frac{1}{2}\sum_{k\ell} M_{k\ell}(\{q\})\dot{q}_k\dot{q}_\ell, \qquad \text{with} \qquad M_{k\ell}(\{q\}) = \sum_j m_j \frac{\partial x_j}{\partial q_k}\frac{\partial x_j}{\partial q_\ell}, \quad (1.12)$$

where $M_{k\ell}$ is known as the **mass matrix**, and is always symmetric but not necessarily diagonal or coordinate independent.

The mass matrix is independent of the $\partial x_j/\partial t$ terms, and we can understand the results we just obtained for it in our two-dimensional example

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[7]It involves quadratic and lower order terms in the velocities, not just quadratic ones.
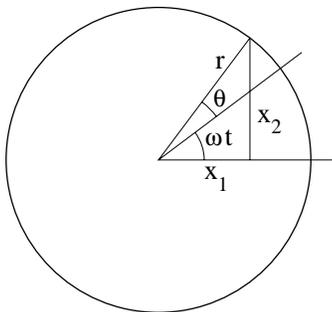
Rotating polar coordinates related to inertial cartesian coordinates.

above,

$$M_{11} = m, \qquad M_{12} = M_{21} = 0, \qquad M_{22} = mr^2,$$

by considering the case without rotation, $\omega = 0$. We can also derive this expression for the kinetic energy in nonrotating polar coordinates by expressing the velocity vector $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ in terms of unit vectors in the radial and tangential directions respectively. The coefficients of these unit vectors can be understood graphically with geometric arguments. This leads more quickly to $\vec{v}^2 = (\dot{r})^2 + r^2(\dot{\theta})^2$, $T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$, and the mass matrix follows. Similar geometric arguments are usually used to find the form of the kinetic energy in spherical coordinates, but the formal approach of (1.12) enables us to find the form even in situations where the geometry is difficult to picture.

It is important to keep in mind that when we view T as a function of coordinates and velocities, these are independent arguments evaluated at a particular moment of time. Thus we can ask independently how T varies as we change x_i or as we change $\dot{x}_i$, each time holding the other variable fixed. Thus the kinetic energy is not a function on the $3n$-dimensional configuration space, but on a larger, $6n$-dimensional space[8] with a point specifying both the coordinates $\{q_i\}$ and the velocities $\{\dot{q}_i\}$.

1.4 Phase Space

If the trajectory of the system in configuration space, $\vec{r}(t)$, is known, the velocity as a function of time, $\vec{v}(t)$ is also determined. As the mass of the particle is simply a physical constant, the momentum $\vec{p} = m\vec{v}$ contains the same information as the velocity. Viewed as functions of time, this gives nothing beyond the information in the trajectory. But at any given time, $\vec{r}$ and $\vec{p}$ provide a complete set of initial conditions, while $\vec{r}$ alone does not. We define **phase space** as the set of possible positions and momenta for the system at some instant. Equivalently, it is the set of possible initial conditions, or the set of possible motions obeying the equations of motion[9]. For a single particle in cartesian coordinates, the six coordinates of phase

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[8]This space is called the tangent bundle to configuration space. For cartesian coordinates it is almost identical to phase space, which is in general the "**cotangent bundle**" to configuration space.

[9]As each initial condition gives rise to a unique future development of a trajectory, there is an isomorphism between initial conditions and allowed trajectories.

space are the three components of $\vec{r}$ and the three components of $\vec{p}$. At any instant of time, the system is represented by a point in this space, called the **phase point**, and that point moves with time according to the physical laws of the system. These laws are embodied in the force function, which we now consider as a function of $\vec{p}$ rather than $\vec{v}$, in addition to $\vec{r}$ and t. We may write these equations as

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{\vec{p}}{m}, \\ \frac{d\vec{p}}{dt} &= \vec{F}(\vec{r}, \vec{p}, t). \end{aligned}$$

Note that these are first order equations, which means that the motion of the point representing the system in phase space is completely determined[10] by where the phase point is. This is to be distinguished from the trajectory in configuration space, where in order to know the trajectory you must have not only an initial point (position) but also its initial time derivative.

1.4.1   Dynamical Systems

We have spoken of the coordinates of phase space for a single particle as $\vec{r}$ and $\vec{p}$, but from a mathematical point of view these together give the coordinates of the phase point in phase space. We might describe these coordinates in terms of a six dimensional vector $\vec{\eta} = (r_1, r_2, r_3, p_1, p_2, p_3)$. The physical laws determine at each point a **velocity function** for the phase point as it moves through phase space,

$$\frac{d\vec{\eta}}{dt} = \vec{V}(\vec{\eta}, t), \tag{1.13}$$

which gives the velocity at which the phase point representing the system moves through phase space. Only half of this velocity is the ordinary velocity, while the other half represents the rapidity with which the momentum is changing, *i.e.* the force. The path traced by the phase point as it travels through phase space is called the **phase curve**.

For a system of n particles in three dimensions, the complete set of initial conditions requires $3n$ spatial coordinates and $3n$ momenta, so phase space is $6n$ dimensional. While this certainly makes visualization difficult, the large

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[10]We will assume throughout that the force function is a well defined continuous function of its arguments.

dimensionality is no hindrance for formal developments. Also, it is sometimes possible to focus on particular dimensions, or to make generalizations of ideas familiar in two and three dimensions. For example, in discussing integrable systems (7.1), we will find that the motion of the phase point is confined to a $3n$-dimensional torus, a generalization of one and two dimensional tori, which are circles and the surface of a donut respectively.

Thus for a system composed of a finite number of particles, the dynamics is determined by the first order ordinary differential equation (1.13), formally a very simple equation. All of the complication of the physical situation is hidden in the large dimensionality of the dependent variable $\vec{\eta}$ and in the functional dependence of the velocity function $V(\vec{\eta}, t)$ on it.

There are other systems besides Newtonian mechanics which are controlled by equation (1.13), with a suitable velocity function. Collectively these are known as **dynamical systems**. For example, individuals of an asexual mutually hostile species might have a fixed birth rate b and a death rate proportional to the population, so the population would obey the **logistic equation**[11] $dp/dt = bp - cp^2$, a dynamical system with a one-dimensional space for its dependent variable. The populations of three competing species could be described by eq. (1.13) with $\vec{\eta}$ in three dimensions.

The dimensionality d of $\vec{\eta}$ in (1.13) is called the **order of the dynamical system**. A d'th order differential equation in one independent variable may always be recast as a first order differential equation in d variables, so it is one example of a d'th order dynamical system. The space of these dependent variables is called the phase space of the dynamical system. Newtonian systems always give rise to an even-order system, because each spatial coordinate is paired with a momentum. For n particles unconstrained in D dimensions, the order of the dynamical system is $d = 2nD$. Even for constrained Newtonian systems, there is always a pairing of coordinates and momenta, which gives a restricting structure, called the symplectic structure[12], on phase space.

If the force function does not depend explicitly on time, we say the system is **autonomous**. The velocity function has no explicit dependance on time, $\vec{V} = \vec{V}(\vec{\eta})$, and is a time-independent vector field on phase space, which we can indicate by arrows just as we might the electric field in ordinary space, or the velocity field of a fluid in motion. This gives a visual indication of

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[11]This is not to be confused with the simpler *logistic map*, which is a recursion relation with the same form but with solutions displaying a very different behavior.

[12]This will be discussed in sections (6.3) and (6.6).

the motion of the system's point. For example, consider a damped harmonic oscillator with $\vec{F} = -kx - \alpha p$, for which the velocity function is

$$\left(\frac{dx}{dt}, \frac{dp}{dt} \right) = \left(\frac{p}{m}, -kx - \alpha p \right).$$

A plot of this field for the undamped ($\alpha = 0$) and damped oscillators is
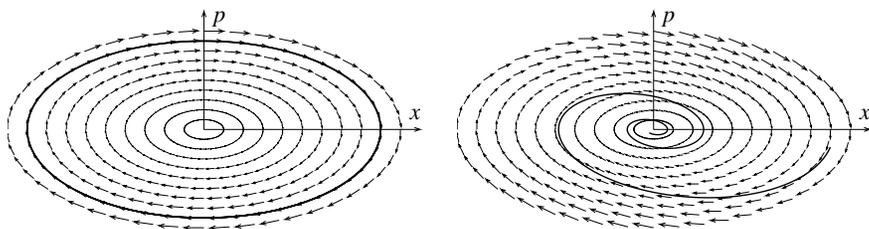

Figure 1.1: Velocity field for undamped and damped harmonic oscillators, and one possible phase curve for each system through phase space.

shown in Figure 1.1. The velocity field is everywhere tangent to any possible path, one of which is shown for each case. Note that qualitative features of the motion can be seen from the velocity field without any solving of the differential equations; it is clear that in the damped case the path of the system must spiral in toward the origin.

The paths taken by possible physical motions through the phase space of an autonomous system have an important property. Because the rate and direction with which the phase point moves away from a given point of phase space is completely determined by the velocity function at that point, if the system ever returns to a point it must move away from that point exactly as it did the last time. That is, if the system at time T returns to a point in phase space that it was at at time $t = 0$, then its subsequent motion must be just as it was, so $\vec{\eta}(T + t) = \vec{\eta}(t)$, and the motion is **periodic** with **period** T. This almost implies that the phase curve the object takes through phase space must be nonintersecting[13].

In the non-autonomous case, where the velocity field is time dependent, it may be preferable to think in terms of extended phase space, a $6n + 1$

[13]An exception can occur at an unstable equilibrium point, where the velocity function vanishes. The motion can just end at such a point, and several possible phase curves can terminate at that point.

dimensional space with coordinates $(\vec{\eta}, t)$. The velocity field can be extended to this space by giving each vector a last component of 1, as $dt/dt = 1$. Then the motion of the system is relentlessly upwards in this direction, though still complex in the others. For the undamped one-dimensional harmonic oscillator, the path is a helix in the three dimensional extended phase space.

Most of this book is devoted to finding analytic methods for exploring the motion of a system. In several cases we will be able to find exact analytic solutions, but it should be noted that these exactly solvable problems, while very important, cover only a small set of real problems. It is therefore important to have methods other than searching for analytic solutions to deal with dynamical systems. Phase space provides one method for finding qualitative information about the solutions. Another approach is numerical. Newton's Law, and more generally the equation (1.13) for a dynamical system, is a set of ordinary differential equations for the evolution of the system's position in phase space. Thus it is always subject to numerical solution given an initial configuration, at least up until such point that some singularity in the velocity function is reached. One primitive technique which will work for all such systems is to choose a small time interval of length Δt, and use $d\vec{\eta}/dt$ at the beginning of each interval to approximate $\Delta \vec{\eta}$ during this interval. This gives a new approximate value for $\vec{\eta}$ at the end of this interval, which may then be taken as the beginning of the next.[14]

[14]This is a very unsophisticated method. The errors made in each step for $\Delta \vec{r}$ and $\Delta \vec{p}$ are typically $\mathcal{O}(\Delta t)^2$. As any calculation of the evolution from time t_0 to t_f will involve a number ($[t_f - t_0]/\Delta t$) of time steps which grows inversely to Δt, the cumulative error can be expected to be $\mathcal{O}(\Delta t)$. In principle therefore we can approach exact results for a finite time evolution by taking smaller and smaller time steps, but in practise there are other considerations, such as computer time and roundoff errors, which argue strongly in favor of using more sophisticated numerical techniques, with errors of higher order in Δt. Increasingly sophisticated methods can be generated which give cumulative errors of order $\mathcal{O}((\Delta t)^n)$, for any n. A very common technique is called fourth-order Runge-Kutta, which gives an error $\mathcal{O}((\Delta t)^5)$. These methods can be found in any text on numerical methods.

As an example, we show the meat of a calculation for the damped harmonic oscillator. This same technique will work even with a very complicated situation. One need only add lines for all the components of the position and momentum, and change the force law appropriately.

This is not to say that numerical solution is a good way to solve this problem. An analytical solution, if it can be found, is almost always preferable, because

```
while (t < tf) {
  dx = (p/m) * dt;
  dp = -(k*x+alpha*p)*dt;
  x = x + dx;
  p = p + dp;
  t = t + dt;
  print t, x, p;
}
```

Integrating the motion, for a damped harmonic oscillator.

- It is far more likely to provide insight into the qualitative features of the motion.

- Numerical solutions must be done separately for each value of the parameters (k, m, α) and each value of the initial conditions (x_0 and p_0).

- Numerical solutions have subtle numerical problems in that they are only exact as $\Delta t \to 0$, and only if the computations are done exactly. Sometimes uncontrolled approximate solutions lead to surprisingly large errors.

Nonetheless, numerical solutions are often the only way to handle a real problem, and there has been extensive development of techniques for efficiently and accurately handling the problem, which is essentially one of solving a system of first order ordinary differential equations.

1.4.2 Phase Space Flows

As we just saw, Newton's equations for a system of particles can be cast in the form of a set of first order ordinary differential equations in time on phase space, with the motion in phase space described by the velocity field. This could be more generally discussed as a d'th order dynamical system, with a phase point representing the system in a d-dimensional phase space, moving

with time t along the velocity field, sweeping out a path in phase space called the phase curve. The phase point $\vec{\eta}(t)$ is also called the state of the system at time t. Many qualitative features of the motion can be stated in terms of the phase curve.

Fixed Points

There may be points $\vec{\eta}_k$, known as **fixed points**, at which the velocity function vanishes, $\vec{V}(\vec{\eta}_k) = 0$. This is a point of equilibrium for the system, for if the system is at a fixed point at one moment, $\vec{\eta}(t_0) = \vec{\eta}_k$, it remains at that point. At other points, the system does not stay put, but there may be sets of states which flow into each other, such as the elliptical orbit for the undamped harmonic oscillator. These are called **invariant sets of states**. In a first order dynamical system[15], the fixed points divide the line into intervals which are invariant sets.

Even though a first-order system is smaller than any Newtonian system, it is worthwhile discussing briefly the phase flow there. We have been assuming the velocity function is a smooth function — generically its zeros will be first order, and near the fixed point η_0 we will have $V(\eta) \approx c(\eta - \eta_0)$. If the constant $c < 0$, $d\eta/dt$ will have the opposite sign from $\eta - \eta_0$, and the system will flow *towards* the fixed point, which is therefore called **stable**. On the other hand, if $c > 0$, the displacement $\eta - \eta_0$ will grow with time, and the fixed point is unstable. Of course there are other possibilities: if $V(\eta) = c\eta^2$, the fixed point $\eta = 0$ is stable from the left and unstable from the right. But this kind of situation is somewhat artificial, and such a system is **structurally unstable**. What that means is that if the velocity field is perturbed by a small smooth variation $V(\eta) \to V(\eta) + \epsilon w(\eta)$, for some bounded smooth function w, the fixed point at $\eta = 0$ is likely to either disappear or split into two fixed points, whereas the fixed points discussed earlier will simply be shifted by order ϵ in position and will retain their stability or instability. Thus the simple zero in the velocity function is **structurally stable**. Note that structual stability is quite a different notion from stability of the fixed point.

In this discussion of stability in first order dynamical systems, we see that generically the stable fixed points occur where the velocity function decreases through zero, while the unstable points are where it increases through zero.

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[15]Note that this is not a one-dimensional Newtonian system, which is a two dimensional $\vec{\eta} = (x, p)$ dynamical system.

Thus generically the fixed points will alternate in stability, dividing the phase line into open intervals which are each invariant sets of states, with the points in a given interval flowing either to the left or to the right, but never leaving the open interval. The state never reaches the stable fixed point because the time $t = \int d\eta/V(\eta) \approx (1/c) \int d\eta/(\eta - \eta_0)$ diverges. On the other hand, in the case $V(\eta) = c\eta^2$, a system starting at η_0 at $t = 0$ has a motion given by $\eta = (\eta_0^{-1} - ct)^{-1}$, which runs off to infinity as $t \to 1/\eta_0 c$. Thus the solution terminates at $t = 1/\eta_0 c$, and makes no sense thereafter. This form of solution is called **terminating motion**.

For higher order dynamical systems, the d equations $V_i(\vec{\eta}) = 0$ required for a fixed point will generically determine the d variables η_j, so the generic form of the velocity field near a fixed point η_0 is $V_i(\vec{\eta}) = \sum_j M_{ij}(\eta_j - \eta_{0j})$ with a nonsingular matrix M. The stability of the flow will be determined by this d-dimensional square matrix M. Generically the eigenvalue equation, a d'th order polynomial in λ, will have d distinct solutions. Because M is a real matrix, the eigenvalues must either be real or come in complex conjugate pairs. For the real case, whether the eigenvalue is positive or negative determines the instability or stability of the flow along the direction of the eigenvector. For a pair of complex conjugate eigenvalues $\lambda = u + iv$ and $\lambda^* = u - iv$, with eigenvectors $\vec{e}$ and $\vec{e}^*$ respectively, we may describe the flow in the plane $\delta\vec{\eta} = \vec{\eta} - \vec{\eta}_0 = x(\vec{e} + \vec{e}^*) + iy(\vec{e} - \vec{e}^*)$, so

$$
\begin{aligned}
\dot{\vec{\eta}} &= M \cdot \delta\vec{\eta} = x(\lambda\vec{e} + \lambda^*\vec{e}^*) + iy(\lambda\vec{e} - \lambda^*\vec{e}^*) \\
&= (ux - vy)(\vec{e} + \vec{e}^*) + (vx + uy)(\vec{e} - \vec{e}^*)
\end{aligned}
$$

so

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{or} \quad \begin{cases} x &= Ae^{ut}\cos(vt + \phi) \\ y &= Ae^{ut}\sin(vt + \phi) \end{cases}.
$$

Thus we see that the motion spirals in towards the fixed point if u is negative, and spirals away from the fixed point if u is positive. Stability in these directions is determined by the sign of the real part of the eigenvalue.

In general, then, stability in each subspace around the fixed point $\vec{\eta}_0$ depends on the sign of the real part of the eigenvalue. If all the real parts are negative, the system will flow from anywhere in some neighborhood of $\vec{\eta}_0$ towards the fixed point, so $\lim_{t\to\infty} \vec{\eta}(t) = \vec{\eta}_0$ provided we start in that neighborhood. Then $\vec{\eta}_0$ is an **attractor** and is a **strongly stable** fixed point. On the other hand, if some of the eigenvalues have positive real parts, there are unstable directions. Starting from a generic point in any neighborhood

of $\vec{\eta}_0$, the motion will eventually flow out along an unstable direction, and the fixed point is considered **unstable**, although there may be subspaces along which the flow may be into $\vec{\eta}_0$. An example is the line $x = y$ in the **hyperbolic fixed point** case shown in Figure 1.2.

Some examples of two dimensional flows in the neighborhood of a generic fixed point are shown in Figure 1.2. Note that none of these describe the fixed point of the undamped harmonic oscillator of Figure 1.1. We have discussed *generic* situations as if the velocity field were chosen arbitrarily from the set of all smooth vector functions, but in fact Newtonian mechanics imposes constraints on the velocity fields in many situations, in particular if there are conserved quantities.
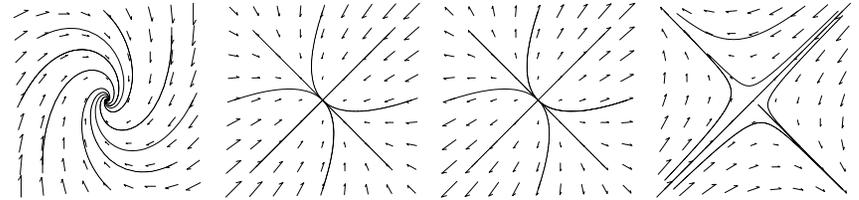
$$
\begin{aligned}
\dot{x} &= -x + y, & \dot{x} &= -3x - y, & \dot{x} &= 3x + y, & \dot{x} &= -x - 3y, \\
\dot{y} &= -2x - y. & \dot{y} &= -x - 3y. & \dot{y} &= x + 3y. & \dot{y} &= -3x - y.
\end{aligned}
$$

Strongly stable spiral point.

Strongly stable fixed point,

Unstable fixed point,

Hyperbolic fixed point,

$$\lambda = -1 \pm \sqrt{2}i. \qquad \lambda = -1, -2. \qquad \lambda = 1, 2. \qquad \lambda = -2, 1.$$

Figure 1.2: Four generic fixed points for a second order dynamical system.

Effect of conserved quantities on the flow

If the system has a conserved quantity $Q(q, p)$ which is a function on phase space only, and not of time, the flow in phase space is considerably changed. This is because the equations $Q(q, p) = K$ gives a set of subsurfaces or contours in phase space, and the system is confined to stay on whichever contour it is on initially. Unless this conserved quantity is a trivial function,

i.e. constant, in the vicinity of a fixed point, it is not possible for all points to flow into the fixed point, and thus it is not strongly stable.

For the case of a single particle in a potential, the total energy $E = p^2/2m + U(\vec{r})$ is conserved, and so the motion of the system is confined to one surface of a given energy. As $\vec{p}/m$ is part of the velocity function, a fixed point must have $\vec{p} = 0$. The vanishing of the other half of the velocity field gives $\nabla U(\vec{r}_0) = 0$, which is the condition for a stationary point of the potential energy, and for the force to vanish. If this point is a maximum or a saddle of U, the motion along a descending path will be unstable. If the fixed point is a minimum of the potential, the region $E(\vec{r}, \vec{p}) < E(\vec{r}_0, 0) + \epsilon$, for sufficiently small ϵ, gives a neighborhood around $\vec{\eta}_0 = (\vec{r}_0, 0)$ to which the motion is confined if it starts within this region. Such a fixed point is called **stable**[16], but it is not strongly stable, as the flow does not settle down to $\vec{\eta}_0$. This is the situation we saw for the undamped harmonic oscillator. For that situation $F = -kx$, so the potential energy may be taken to be

$$U(x) = \int_x^0 -kx \; dx = \frac{1}{2}kx^2,$$

and so the total energy $E = p^2/2m + \frac{1}{2}kx^2$ is conserved. The curves of constant E in phase space are ellipses, and each motion orbits the appropriate ellipse, as shown in Fig. 1.1 for the undamped oscillator. This contrasts to the case of the damped oscillator, for which there is no conserved energy, and for which the origin is a *strongly* stable fixed point.

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[16]A fixed point is **stable** if it is in arbitrarily small neighborhoods, each with the property that if the system is in that neighborhood at one time, it remains in it at all later times.

As an example of a conservative system with both stable and unstable fixed points, consider a particle in one dimension with a cubic potential $U(x) = ax^2 - bx^3$, as shown in Fig. 1.3. There is a stable equilibrium at $x_s = 0$ and an unstable one at $x_u = 2a/3b$. Each has an associated fixed point in phase space, an **elliptic fixed point** $\eta_s = (x_s, 0)$ and a hyperbolic fixed point $\eta_u = (x_u, 0)$. The velocity field in phase space and several possible orbits are shown. Near the stable equilibrium, the trajectories are approximately ellipses, as they were for the harmonic oscillator, but for larger energies they begin to feel the asymmetry of the potential, and the orbits become egg-shaped.
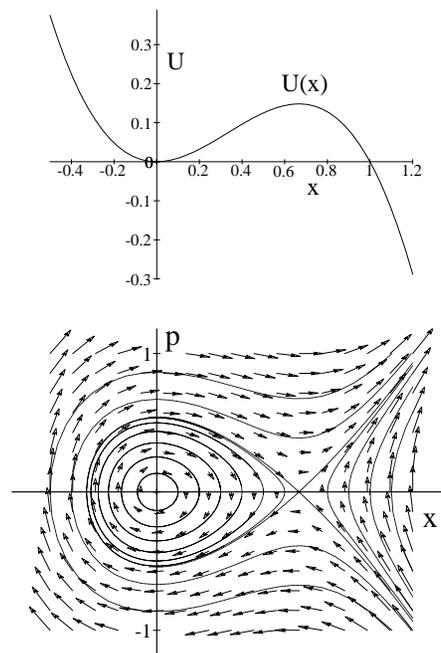

Figure 1.3. Motion in a cubic potential.

If the system has total energy precisely $U(x_u)$, the contour line crosses itself. This contour actually consists of three separate orbits. One starts at $t \to -\infty$ at $x = x_u$, completes one trip though the potential well, and returns as $t \to +\infty$ to $x = x_u$. The other two are orbits which go from $x = x_u$ to $x = \infty$, one incoming and one outgoing. For $E > U(x_u)$, all the orbits start and end at $x = +\infty$. Note that generically the orbits deform continuously as the energy varies, but at $E = U(x_u)$ this is not the case — the character of the orbit changes as E passes through $U(x_u)$. An orbit with this critical value of the energy is called a **separatrix**, as it separates regions in phase space where the orbits have different qualitative characteristics.

Quite generally hyperbolic fixed points are at the ends of separatrices. In our case the contour $E = U(x_u)$ consists of four invariant sets of states, one of which is the point η_u itself, and the other three are the orbits which are

the disconnected pieces left of the contour after removing η_u.

Exercises

1.1 (a) Find the potential energy function $U(\vec{r})$ for a particle in the gravitational field of the Earth, for which the force law is $\vec{F}(\vec{r}) = -GM_E m\vec{r}/r^3$.
(b) Find the escape velocity from the Earth, that is, the minimum velocity a particle near the surface can have for which it is possible that the particle will eventually coast to arbitrarily large distances without being acted upon by any force other than gravity. The Earth has a mass of 6.0×10^{24} kg and a radius of 6.4×10^6 m. Newton's gravitational constant is $6.67 \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$.

1.2 In the discussion of a system of particles, it is important that the particles included in the system remain the same. There are some situations in which we wish to focus our attention on a set of particles which changes with time, such as a rocket ship which is emitting gas continuously. The equation of motion for such a problem may be derived by considering an infinitesimal time interval, $[t, t+\Delta t]$, and choosing the system to be the rocket with the fuel still in it at time t, so that at time $t + \Delta t$ the system consists of the rocket with its remaining fuel *and also* the small amount of fuel emitted during the infinitesimal time interval.
Let $M(t)$ be the mass of the rocket and remaining fuel at time t, assume that the fuel is emitted with velocity $\vec{u}$ with respect to the rocket, and call the velocity of the rocket $\vec{v}(t)$ in an inertial coordinate system. If the external force on the rocket is $\vec{F}(t)$ and the external force on the infinitesimal amount of exhaust is infinitesimal, the fact that $F(t)$ is the rate of change of the total momentum gives the equation of motion for the rocket.
(a) Show that this equation is

$$M\frac{d\vec{v}}{dt} = \vec{F}(t) + \vec{u}\frac{dM}{dt}.$$

(b) Suppose the rocket is in a constant gravitational field $\vec{F} = -Mg\hat{e}_z$ for the period during which it is burning fuel, and that it is fired straight up with constant exhaust velocity $(\vec{u} = -u\hat{e}_z)$, starting from rest. Find $v(t)$ in terms of t and $M(t)$.
(c) Find the maximum fraction of the initial mass of the rocket which can escape the Earth's gravitational field if $u = 2000$m/s.

1.3 For a particle in two dimensions, we might use polar coordinates (r, θ) and use basis unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ in the radial and tangent directions respectively to describe more general vectors. Because this pair of unit vectors differ from point

to point, the $\hat{e}_r$ and $\hat{e}_\theta$ along the trajectory of a moving particle are themselves changing with time.
(a) Show that

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta, \qquad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r.$$

(b) Thus show that the derivative of $\vec{r} = r\hat{e}_r$ is

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta,$$

which verifies the discussion of Sec. (1.3.4).
(c) Show that the derivative of the velocity is

$$\vec{a} = \frac{d}{dt}\vec{v} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta.$$

(d) Thus Newton's Law says for the radial and tangential components of the force are $F_r = \hat{e}_r \cdot F = m(\ddot{r} - r\dot{\theta}^2)$, $F_\theta = \hat{e}_\theta \cdot F = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$. Show that the generalized forces are $Q_r = F_r$ and $Q_\theta = rF_\theta$.

1.4 Analyze the errors in the integration of Newton's Laws in the simple Euler's approach described in section 1.4.1, where we approximated the change for x and p in each time interval Δt between t_i and t_{i+1} by $\dot{x}(t) \approx \dot{x}(t_i)$, $\dot{p}(t) \approx F(x(t_i), v(t_i))$. Assuming F to be differentiable, show that the error which accumulates in a finite time interval T is of order $(\Delta t)^1$.

1.5 Write a simple program to integrate the equation of the harmonic oscillator through one period of oscillation, using Euler's method with a step size Δt. Do this for several Δt, and see whether the error accumulated in one period meets the expectations of problem 1.4.

1.6 Describe the one dimensional phase space for the logistic equation $\dot{p} = bp - cp^2$, with $b > 0, c > 0$. Give the fixed points, the invariant sets of states, and describe the flow on each of the invariant sets.

1.7 Consider a pendulum consisting of a mass at the end of a massless rod of length L, the other end of which is fixed but free to rotate. Ignore one of the horizontal directions, and describe the dynamics in terms of the angle θ between the rod and the downwards direction, without making a small angle approximation.
(a) Find the generalized force Q_θ and find the conserved quantity on phase space.
(b) Give a sketch of the velocity function, including all the regions of phase space. Show all fixed points, separatrices, and describe all the invariant sets of states. [Note: the variable θ is defined only modulo 2π, so the phase space is the

Cartesian product of an interval of length 2π in θ with the real line for p_θ. This can be plotted on a strip, with the understanding that the left and right edges are identified. To avoid having important points on the boundary, it would be well to plot this with $\theta \in [-\pi/2, 3\pi/2]$.

1.8 Consider again the pendulum of mass m on a massless rod of length L, with motion restricted to a fixed vertical plane, with θ, the angle made with the downward direction, the generalized coordinate. Using the fact that the energy E is a constant,
(a) Find $d\theta/dt$ as a function of θ.
(b) Assuming the energy is such that the mass comes to rest at $\theta = \pm\theta_0$, find an integral expression for the period of the pendulum.
(c) Show that the answer is $4\sqrt{\frac{L}{g}}K(\sin^2(\theta_0/2))$, where

$$K(m) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$$

is the complete elliptic integral of the first kind.
(Note: the circumference of an ellipse is $4aK(e^2)$, where a is the semi-major axis and e the eccentricity.)
(d) Show that $K(m)$ is given by the power series expansion

$$K(m) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 m^n,$$

and give an estimate for the ratio of the period for $\theta_0 = 60°$ to that for small angles.

1.9 As mentioned in the footnote in section 1.3, a current i_1 flowing through a wire segment $d\vec{s}_1$ at $\vec{s}_1$ exerts a force

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} i_1 i_2 \frac{d\vec{s}_2 \times (d\vec{s}_1 \times \vec{r})}{|r|^3}$$

on a current i_2 flowing through a wire segment $d\vec{s}_2$ at $\vec{s}_2$, where $\vec{r} = \vec{s}_2 - \vec{s}_1$.
(a) Show, as stated in that footnote, that the sum of this force and its Newtonian reaction force is

$$\vec{F}_{12} + \vec{F}_{21} = \frac{\mu_0}{4\pi} \frac{i_1 i_2}{|r|^3} \left[d\vec{s}_1(d\vec{s}_2 \cdot \vec{r}) - d\vec{s}_2(d\vec{s}_1 \cdot \vec{r})\right],$$

which is not generally zero.
(b) Show that if the currents each flow around closed loops, the total force $\oint \oint F_{12} + F_{21}$ vanishes.
[Note: Eq. (A.7) of appendix (A.1) may be useful, along with Stokes' theorem.]