Physics 504, Lecture 26
May 2, 2011

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1 Electromagnetic Interactions

In this course we are dealing with electromagnetic fields and charged particles, which, of course, interact with each other as specified by Maxwell's laws and the Lorentz force equation. We first discussed the dynamics of electromagnetic fields with certain boundary conditions representing the simplified effect of charges, for example with ideally conducting wave guides or linear dielectric materials. We also considered the fields produced by specified particle motions. Inversely, we considered the charged particle motions is specified external fields. We are able to get quite good descriptions of many important contexts using this simplified approach, though in principle we should be considering how interactions of the fields and particles act on each other on a continuous basis. When external fields cause particles to accelerate, they produce additional fields, which of course changes the motion of the particles. As an example, consider a charged particle moving in a magnetostatic field. We say that the kinetic energy is conserved because the Lorentz force of a magnetic field is perpendicular to the velocity, but if the particle is accelerated it must be radiating, and then the energy of radiation must be coming from the kinetic energy of the particle, and it must, in fact, be slowing down.

The reason that we were able to get so many useful results in many contexts by considering the interactions to go in only one direction is that often the back-reaction is small. For example, for a nonrelativistic particle we saw that the power radiated is $P = 2e^2a^2/3c^3$. If we examine the motion over a time interval T for which the acceleration remains roughly in the same direction, the energy radiated is $E_{\rm rad} \approx 2e^2a^2T/3c^3$ while the velocity will change by $\approx aT$ with a kinetic energy comparable to $E_0 = \frac{1}{2}mv^2 \sim m(aT)^2$, so radiative effects will be small if

$$\frac{2e^2a^2T}{3c^3} \ll ma^2T^2 \Longrightarrow T \gg \tau := \frac{2e^2}{3mc^3} \qquad \left(\frac{e^2}{6\pi\epsilon_0 mc^3} \text{ in SI units}\right).$$

The characteristic time for an electron is

$$\tau = \frac{2e^2}{3m_e c^3} = 6.26 \times 10^{-24}\ \mathrm{s}, \qquad c\tau = 1.88\ \text{femtometers},$$

so for macroscopic phenomena this is pretty much okay. Of course if the particle would have been in periodic motion without the radiation, and if we wait long enough, the radiated energy will degrade the orbit, but in any one period the radiated energy is roughly

$$E_{\rm rad} \sim \frac{2e^2}{3c^3}\omega^4 r^2 \frac{2\pi}{\omega} \quad \text{compared to} \quad E_0 \sim m\omega^2 r^2,$$

so the radiated power has only an adiabatic effect provided $\omega\tau \ll 1$.

2 Radiative Reaction

Of course over a long time the adiabatic effect will be significant, so it would be good to have a description of a nonrelativistic particle's motion which includes the back reaction in an average sense. If we had no radiation the particle would obey Newton: $m\dot{\vec{v}} = \vec{F}_{\rm ext}$, but it would also radiate power at a rate $P(t) = 2e^2(\dot{\vec{v}})^2/3c^3$, and therefore it would feel a damping force $\vec{F}_{\rm rad}$ which does negative work on it:

$$-\int_{t_1}^{t_2} P(t)\,dt = \int_{t_1}^{t_2} \vec{F}_{\rm rad}\vec{v}(t)\,dt = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \dot{\vec{v}}\cdot\dot{\vec{v}}\,dt = \frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{\vec{v}}\cdot\vec{v}\,dt - \frac{2e^2}{3c^3}\dot{\vec{v}}\cdot\vec{v}\Big|_{t_1}^{t_2}.$$

If we have an excuse for throwing away the endpoint terms, say because the particle is in quasi-periodic motion or that we can pick times t_1 and t_2 for which $\dot{\vec{v}}\cdot\vec{v} = 0$, we can say that the radiative damping force is

$$\vec{F}_{\rm rad} = \frac{2e^2}{3c^3}\ddot{\vec{v}} = m\tau\ddot{\vec{v}}.$$

This gives an equation of motion, known as the Abraham-Lorentz equation, which involves second order time derivatives of $\vec{v}$, which is to say third order derivatives of position, that is, the jerk, which violates the usual rules for writing down laws of motion. And for good reason, because the additional solution which the extra order of derivative provides allows for a particle with no external forces to take off by itself, $x(t) = x_0 e^{t/\tau}$, with the particle speeding up indefinitely without any external source of energy! On the other hand, if we have solutions which are small perturbations on the equation without the damping force, we can use these to describe the motion.

This suggests using the undamped equation to evaluate the radiation damping term, $\vec{F}_{\rm rad} = \tau dF_{\rm ext}/dt$

$$m\dot{\vec{v}} = F_{\rm ext} + \tau\left[\frac{\partial F_{\rm ext}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)F_{\rm ext}(\vec{x},t)\right]. \qquad (1)$$

Consider a particle in a central force field with potential $U(r)$, which in the absence of radiative damping would have a conserved energy E and a conserved angular momentum $\vec{L}$, a force $\vec{F}_{\rm ext} = -\dfrac{dU}{dr}\dfrac{\vec{r}}{r}$ and an acceleration $\dot{\vec{v}} = -\frac{1}{m}\frac{\vec{r}}{r}\frac{dU}{dr}$. The particle will lose energy at a rate $P(t)$, of course, so

$$\frac{dE}{dt} = -\frac{2e^2}{3c^3}(\dot{\vec{v}})^2 = -\frac{2e^2}{3m^2c^3}\left(\frac{dU}{dr}\right)^2 = -\frac{\tau}{m}\left(\frac{dU}{dr}\right)^2.$$

The rate of change of the angular momentum is

$$\frac{d\vec{L}}{dt} = \vec{r}\times m\dot{\vec{v}} = \vec{r}\times\left[-\frac{\vec{r}}{r}\frac{dU}{dr} - \tau(\vec{v}\cdot\vec{\nabla})\left(\frac{\vec{r}}{r}\frac{dU}{dr}\right)\right].$$

The first term contains $\vec{r}\times\vec{r}$ and vanishes. So does the term where the gradient acts on dU/dr, as that also contains $\vec{r}\times\vec{r}$. Using $\vec{v}\cdot\vec{\nabla}\hat{e}_r = \vec{v}/r - \vec{r}(\vec{r}\cdot\vec{v})/r^2$, we find

$$\frac{d\vec{L}}{dt} = -\tau\vec{r}\times\left(\frac{\vec{v}}{r} - \vec{r}\,\frac{\vec{r}\cdot\vec{v}}{r^2}\right)\frac{dU}{dr} = \frac{-\tau}{m}\vec{L}\,\frac{1}{r}\frac{dU}{dr}.$$

As we expect that the damping terms have a small effect over one almost-closed orbit, we can consider the averages over an orbit,

$$\begin{aligned}\left\langle\frac{dE}{dt}\right\rangle &= -\frac{\tau}{m}\left\langle\left(\frac{dU}{dr}\right)^2\right\rangle,\\ \left\langle\frac{d\vec{L}}{dt}\right\rangle &= -\frac{\tau}{m}\vec{L}\left\langle\frac{1}{r}\frac{dU}{dr}\right\rangle.\end{aligned}$$

In general the orbit of the motion in the spherical potential, ignoring the radiation damping, is determined by E and $\vec{L}$, up to a rotation, so the values of $\left\langle \left(\frac{dU}{dr}\right)^2 \right\rangle$ and $\left\langle \frac{1}{r}\frac{dU}{dr} \right\rangle$ can be evaluated as functions of E and L, giving a pair of ordinary differential equations which determine how the orbit decays with time. The classical calculation of what would happen classically to an electron in a Bohr orbit in hydrogen is informative: according to Problem 16.2, the electron in the ground state should spiral into the nucleus after a time

$$t = \frac{a_0^3}{9c^3\tau^2} = 15 \text{ picoseconds!}$$

Of course quantum mechanics prevents this from happening. The classical calculation can, however, give reasonable results if you ask for the decay time for a decay $\ell \to \ell - 1$ for large values of ℓ, according to Bohr's correspondence principle, as you considered for homework.

We will skip sections §16.3–16.6

3 Line Width

To work a simpler example, consider a charge on a one-dimensional harmonic oscillator with a force constant $k = m\omega_0^2$, that is, $F_{\text{ext}} = -m\omega_0^2 x$. Assuming $\omega_0\tau \ll 1$, we can use (1), so we have

$$m\dot{v} = -m\omega_0^2(x + \tau v) \to m\ddot{x} + m\omega_0^2\tau\dot{x} + m\omega_0^2 x = 0.$$

This linear ODE with constant coefficients has two solutions $x(t) = x_0 e^{-\alpha t}$ with $\alpha^2 - \tau\omega_0^2\alpha + \omega_0^2 = 0$, or

$$\alpha = \frac{1}{2}\tau\omega_0^2 \pm i\omega_0\sqrt{1 - (\tau\omega_0/2)^2} \approx \frac{1}{2}\tau\omega_0^2 \pm i(\omega_0 - \tau^2\omega_0^3/8).$$

The real part of this is the decay constant $\Gamma/2$, while the imaginary part is the angular frequency, slightly shifted by the damping,

$$\omega = \omega_0 + \Delta\omega, \qquad \text{with } \Delta\omega = -\tau^2\omega_0^3/8.$$

Thus if the oscillator is set going at time zero, it will emit radiation proportional to $(\ddot{x}(t))^2$, which will not have a pure frequency, but rather the distribution of frequencies of the amplitude of emitted radiation is

$$E(\omega) \propto \int_0^\infty e^{-\alpha t} e^{i\omega t} dt = \frac{1}{\alpha - i\omega},$$

whose absolute square give the power spectrum

$$\frac{dI(\omega)}{d\omega} = A\frac{1}{(\Gamma/2)^2 + (\omega - \omega_0 - \Delta\omega)^2},$$

which is called the "resonant line shape" or Lorentzian. The total energy radiated is

$$\begin{aligned} I_0 &= A\int_0^\infty d\omega \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0 - \Delta\omega)^2} \\ &= \frac{2A}{\Gamma}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{2(\omega_0 + \Delta\omega)}{\Gamma}\right)\right] \to \frac{2\pi A}{\Gamma} \end{aligned}$$

where the last expression assumed $\Gamma \ll \omega_0$ as $\omega_0\tau \ll 1$.

In terms of wavelengths, the classical line width is

$$\Delta\lambda = \Gamma|d\lambda/d\omega| = 2\pi c\Gamma/\omega_0^2 = 2\pi c\tau = 1.18 \times 10^{-14} \text{ m}.$$

Quantum mechanically there are oscillator strength factors, but the order of magnitude is correct, so $\Gamma/\omega_0 \sim 10^{-8}$ for optical transitions in atoms, justifying our assumption that $\omega_0\tau \ll 1$.

Jackson points out that the level shift, classically proportional to $\omega_0^3\tau^2$ is not correct quantum mechanically, because of the Lamb shift, an effect of vacuum fluctuations.

4 Scattering by an Oscillator

We have just seen how a charged oscillator radiates away its energy, so now let's turn to how it scatters light. We already considered a free charged particle, with Thomson scattering. Let us assume now the electron is bound by a spherically symmetric spring with spring constant $m\omega_0^2$ as before. In the presence of an incoming electric field, the force on it (assuming it remains non-relativistic) is $\vec{F}_{\text{ext}} = -m\omega_0^2\vec{x} + e\vec{\epsilon}E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}$. From (1) we have

$$m\dot{\vec{v}} = -m\omega_0^2\vec{x} + e\vec{\epsilon}E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t} - \tau m\omega_0^2\dot{\vec{x}} - i\omega\left(\tau - \vec{v}\cdot\vec{k}\right)e\vec{\epsilon}E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}.$$

We are dropping terms proportional to $\vec{v}E_0$ (we didn't consider the magnetic field either) so we have

$$\ddot{\vec{x}} + \Gamma_t\dot{\vec{x}} + \omega_0^2\vec{x} = \frac{eE_0}{m}\vec{\epsilon}(1 - i\omega\tau)e^{i\vec{k}\cdot\vec{x}-i\omega t},$$

where Γ should be $\tau\omega_0^2$, but we will throw in an additional unspecified damping Γ' due to "other dissipative processes", which will be left to a course in quantum atomic physics. So $\Gamma_t = \tau\omega_0^2 + \Gamma'$. Here we are looking for a steady state solution to this inhomogeneous linear equation, rather than the decay of the homogeneous one, and it is

$$\vec{x}(t) = \frac{eE_0}{m}\vec{\epsilon}\frac{(1 - i\omega\tau)e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\Gamma_t}.$$

Larmor tells us the power into $d\Omega$ with polarization $\vec{\epsilon}'$ is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2}\frac{e^2}{4\pi c^3}\left|\vec{\epsilon}'\cdot\left(\hat{n}\times(\hat{n}\times\ddot{\vec{x}})\right)\right|^2 = \frac{e^2}{8\pi c^3}\left|\vec{\epsilon}'\cdot\ddot{\vec{x}}\right|^2 \\ &= \frac{e^2}{8\pi c^3}\left(\frac{eE_0}{m}\right)^2\left|\frac{(1-i\omega\tau)\omega^2}{\omega_0^2 - \omega^2 - i\omega\Gamma_t}\right|^2\left|\vec{\epsilon}'\cdot\vec{\epsilon}\right|^2.\end{aligned}$$

Dividing by the incoming flux density $cE_0^2/8\pi$, we get the cross section

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m^2c^4}\frac{(1+\omega^2\tau^2)\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma_t^2}\left|\vec{\epsilon}'\cdot\vec{\epsilon}\right|^2.$$

We can drop the $\omega^2\tau^2$ compared to 1. To calculate the total cross section, as for the Thomson cross section, we have $\left|\vec{\epsilon}'\cdot\vec{\epsilon}\right|^2 \to 8\pi/3$, so

$$\sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2c^4}\frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma_t^2}.$$

Writing this in terms of the radiation damping width $\Gamma = \omega_0^2\tau = 2e^2\omega_0^2/3mc^3$ and the resonant wavelength $\lambda := 2\pi c/\omega_0$,

$$\sigma_T = \frac{3}{2\pi}\lambda^2\frac{\omega^4\Gamma^2/\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma_t^2}.$$

At low frequencies we have ω^4 behavior, as predicted by Rayleigh's law , and at high frequencies $\sigma_T \to 6\pi(c\tau)^2 = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$, the Thomson cross section. This makes sense, in that if the incoming frequency is much higher than the resonant frequency, the electron doesn't realize it is not free. The strong peak at the resonant frequency $\omega = \omega_0$ is called **resonance fluorescence**.