Physics 504, Lecture 14
March 21, 2011

1 The Optical Theorem

The optical theorem relates the scattering amplitude in the forward direction to the total scattering cross section. It may be familiar from quantum mechanics, where it is related to conservation of probability, but it first arose in optics, hence its name.

To follow Jackson's presentation in §10.11 requires following sections in which he gives details of diffraction and scattering that we are skipping. I will present an alternate argument which doesn't require such details.

Consider the scattering by a scatterer of finite size of an incident plane wave

$$\begin{aligned}\vec{E}_i &= E_0\,\vec{\epsilon}_i\,e^{i\vec{k}_i\cdot\vec{x}-i\omega t}\\ \vec{B}_i &= \frac{1}{\omega}\vec{k}_i\times\vec{E}_i=\frac{1}{\omega}\vec{k}_i\times\vec{\epsilon}_i\,E_0\,e^{i\vec{k}_i\cdot\vec{x}-i\omega t}\end{aligned}$$

The scattered fields will also have a $e^{-i\omega t}$ dependence, assuming the scatterer responds linearly in a time-invariant fashion, so we will drop all such factors.

The scattered wave is proportional to the incident wave and is given at large distances by the *scattering amplitude* $\vec{f}(\vec{k},\vec{k}_i)$ as[1]

$$\begin{aligned}\vec{E}_s(\vec{x}) &= \frac{e^{ikr}}{r}\vec{f}(\vec{k},\vec{k}_i)E_0\\ \vec{B}_s(\vec{x}) &= \frac{1}{\omega}\vec{k}\times\vec{E}_s=\frac{e^{ikr}}{\omega r}E_0\,\vec{k}\times\vec{f}(\vec{k},\vec{k}_i).\end{aligned}$$

The total fields are $\vec{E}=\vec{E}_i+\vec{E}_s$ and $\vec{B}=\vec{B}_i+\vec{B}_s$.

The total power removed from the incident beam must be the incident power flux times the total cross section, which includes both the absorption cross section and the scattering cross section. If we measure the power far

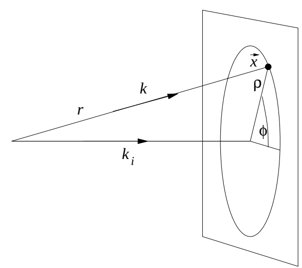
[1] Here $\vec{k}$ is in the direction of $\vec{r}$, and we are looking only at "elastic scattering" with $k=|k|=|k_i|$. It would probably be better to define f as a tensor multiplying $\vec{\epsilon}_i E_0$, but we will not need that generality. Though we haven't made it explicit, $\vec{f}$ depends on $\vec{\epsilon}_i$.

downfield from the scatterer, "behind" the scatterer, we must find that the scatterer has reduced that power by the appropriate amount. If we take $\vec{k}_i$ in the z direction, and measure the power crossing a plane of constant large z, the power flux is given by

$$P = \text{ Re } \int_A \vec{S} \cdot \hat{e}_z \, dx \, dy = \frac{1}{2\mu_0} \int \rho \, d\rho \, d\phi \text{ Re } \left[\left(\vec{E}_i + \vec{E}_s \right) \times \left(\vec{B}_i^* + \vec{B}_s^* \right) \right]_z .$$

The power that would have arrived without the scatterer is the term without the scattered fields. As these scattered fields fall off as $1/r$ at large r, we can ignore the term with two scattered fields, and thus the power removed from the beam is $-\Delta P$, where

$$\begin{aligned} \Delta P &= \frac{1}{2\mu_0} \int \rho \, d\rho \, d\phi \text{ Re } \left[\vec{E}_i \times \vec{B}_s^* + \vec{E}_s \times \vec{B}_i^* \right]_z \\ &= \frac{|E_0|^2}{2\omega\mu_0} \int \frac{\rho}{r} \, d\rho \, d\phi \\ &\qquad \text{Re} \left[\vec{\epsilon}_i \times \left(\vec{k} \times \vec{f}^*(\vec{k}, \vec{k}_i) \right) e^{-ikr + i\vec{k}_i \cdot \vec{x}} \right. \\ &\qquad \left. + e^{ikr - i\vec{k}_i \cdot \vec{x}} \vec{f}(\vec{k}, \vec{k}_i) \times \left(\vec{k}_i \times \vec{\epsilon}_i^{\,*} \right) \right]_z \end{aligned}$$



The contribution for very large values of z will come from $\rho \sim \sqrt{z}$ so the angle goes to zero, $\vec{k} = \vec{k}_i$, $kr - \vec{k}_i \cdot \vec{x} = k(r - z) = k(\sqrt{z^2 + \rho^2} - z) \approx k\rho^2/2z$, so[2]

$$\int \rho \, d\rho \, d\phi \frac{e^{ikr - i\vec{k}_i \cdot \vec{x}}}{\sqrt{z^2 + \rho^2}} \approx \frac{2\pi}{z} \int_0^\infty \rho \, d\rho \, e^{ik\rho^2/2z} = \frac{2\pi}{z} \int_0^\infty du \, e^{iku/z} = i\frac{2\pi}{k}.$$

Thus

$$\begin{aligned} \Delta P &= \frac{\pi \, |E_0^2|}{\omega\mu_0 k} \text{ Re } \Big(-i\vec{\epsilon}_i \cdot \vec{f}^*(\vec{k}_i, \vec{k}_i)\vec{k} + i\vec{\epsilon}_i \cdot \vec{k}\vec{f}^*(\vec{k}_i, \vec{k}_i) \\ &\qquad + i\vec{\epsilon}_i^{\,*} \cdot \vec{f}(\vec{k}_i, \vec{k}_i)\vec{k}_i - i(\vec{k}_i \cdot \vec{f}(\vec{k}_i, \vec{k}_i))\vec{\epsilon}_i^{\,*} \Big)_z \\ &= \frac{\pi \, |E_0^2|}{\omega\mu_0} \text{ Re } \left(-i\vec{\epsilon}_i \cdot \vec{f}^*(\vec{k}, \vec{k}_i) + 0 + i\vec{\epsilon}_i^{\,*} \cdot \vec{f}(\vec{k}_i, \vec{k}_i) - 0 \right) \\ &= -\frac{2\pi \, |E_0^2|}{\omega\mu_0} \text{ Im } \left(\vec{\epsilon}_i^{\,*} \cdot \vec{f}(\vec{k}_i, \vec{k}_i) \right). \end{aligned}$$

[2]Pesky terms like $e^{ik\infty}$ will be dropped with some excuse like k having an infinitesimal positive imaginary part.

The power flux in the incident beam is

$$\frac{1}{2\mu_0}\ \mathrm{Re}\ (\vec{E}_i \times \vec{B}_i^*)_z = \frac{|E_0|^2}{2\omega\mu_0}\ \mathrm{Re}\ \left(\vec{\epsilon}_i \times \left(\vec{k} \times \vec{\epsilon}_i\right)\right)_z = \frac{|E_0|^2 k}{2\omega\mu_0}$$

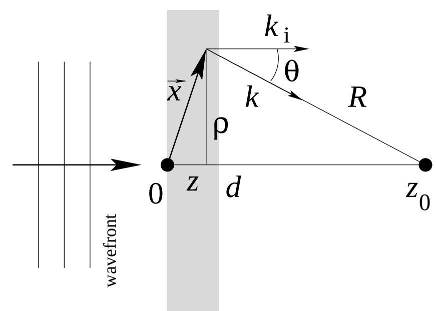
so the total cross section must be

$$\sigma_{\mathrm{Tot}} = \frac{-2\Delta P\omega\mu_0}{|E_0|^2 k} = \frac{4\pi}{k}\ \mathrm{Im}\ \left(\vec{\epsilon}_i{}^* \cdot \vec{f}(\vec{k}_i, \vec{k}_i)\right).$$

This is the optical theorem.

1.1 Index of Refraction

I mentioned that the forward scattering is related to the index of refraction. To see this, consider a thin slab of scattering material, of number density N, thickness d, with an incident wave $\vec{E}_i = E_0\vec{\epsilon}_i e^{i\vec{k}_i\cdot\vec{x}}$, with $\vec{k}_i = k\hat{e}_z$ in the z direction, normal to the surfaces of the slab. Let us observe the field at a large distance z_0. Each d^3x in the slab has Nd^3x scatterers contributing a scattered wave



$$d\vec{E}_s = \frac{e^{ikR}}{R}\vec{f}(k,\theta,\phi;\vec{k}_i)E_0 e^{i\vec{k}_i\cdot\vec{x}} N d^3x$$

$$\vec{E}_s = NE_0\int_0^d dz e^{ikz}\int_0^{2\pi} d\phi \int_0^\infty \rho\, d\rho\, \frac{e^{ikR}}{R}\vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)$$

As $R^2 = \rho^2 + (z_0 - z)^2$, $\rho\, d\rho = R\, dR$, so

$$\begin{aligned}
&\int_0^\infty \rho\, d\rho\, \frac{e^{ikR}}{R}\vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right) \\
&= \int_{|z_0-z|}^\infty dR\, e^{ikR}\vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right) \\
&= \frac{1}{ik}\ e^{ikR}\vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)\Big|_{R=|z_0-z|}^\infty \\
&\qquad -\frac{1}{ik}\int_{|z_0-z|}^\infty e^{ikR}\, dR \frac{d}{dR}\vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)
\end{aligned}$$

where we integrated by parts for the last expression. The last term is

$$\frac{1}{ik}\int_{|z_0-z|}^{\infty} e^{ikR}\, dR \frac{z_0-z}{R^2}\frac{d}{d\cos\theta}\vec{f}\,(k,\theta,\phi;k\hat{e}_z)$$

which, provided the indicated derivative is not singular, falls off like $1/R$. Dropping that term, and the $e^{ik\infty}$ limit from the first term, we have

$$\begin{aligned}\vec{E}_s &= i\frac{NE_0}{k}\int_0^d dz e^{ikz}\int_0^{2\pi} d\phi e^{ik(z_0-z)}\vec{f}\,(k,0,\phi;k\hat{e}_z)\\ &= 2\pi i\frac{NE_0 d}{k}e^{ikz_0}\vec{f}\,(k,0,0;k\hat{e}_z)\end{aligned}$$

Thus the total electric field at points far beyond the slab is

$$\vec{E}(\vec{x}) = E_0 e^{ikz}\left(\vec{\epsilon}_i + \frac{2\pi i N d}{k}\vec{f}(k,0)\right),$$

which is a plane wave with a shifted phase, amplitude and polarization, but is still an exact solution of the free space wave equation. Therefore this expression should hold right up to the edge of the slab. If we project on the original polarization, we see the effect of a thickness dz of material on $\vec{\epsilon}_i^{\;*}\cdot\vec{E}$ is to multiply it by $1+2\pi i k^{-1}N\vec{\epsilon}_i^{\;*}\cdot\vec{f}(k,0)dz$, so integrating this effect for a larger distance would give

$$\vec{\epsilon}_i^{\;*}\cdot\vec{E}(\vec{x}) = e^{2\pi i k^{-1}N\vec{\epsilon}_i^{\;*}\cdot\vec{f}(k,0)z}E_0 e^{ikz},$$

that is, we have the vacuum value k replaced by nk, where the index of refraction n is given by

$$n = 1+\frac{2\pi N\vec{\epsilon}_i^{\;*}\cdot\vec{f}(k,0)}{k^2}.$$

Thus we see that the index of refraction is given by the forward scattering amplitude.

Jackson makes some comments about improving on the assumptions, in particular that the field experienced by each scatterer is unaffected by the others. He tells us the principal effect of fixing this is to evaluate the scattering cross section at the wavenumber suitable for the dielectric rather than for free space.

One effect we can certainly handle is the absorption — as the forward scattering has an imaginary part which gives the total scattering, and the optical theorem relates that to the absorption from the beam, and hence the imaginary part of the $k = nk_i = \text{ Re } k + \frac{i}{2}\alpha k_i$, so

$$\alpha = N\sigma_{\text{tot}} = \frac{4\pi N}{k} \text{ Im } \left(\vec{\epsilon}_i^{\,*} \cdot \vec{f}(\vec{k}, \vec{k})\right).$$

Jackson also warns us that the optical theorem requires the full, correct scattering amplitude f, and that various approximations for f may give an imaginary part inconsistent with the total cross section which depends on $|f^2|$. For a small loss-less dielectric sphere we found a real scattering amplitude because ϵ_r is real, which would give zero for σ_{tot} by the optical theorem, which is clearly wrong. But in section §7.10D you discussed the Kramers-Kronig relation, in which analyticity based on causality gave an integral relationship giving the real part of ϵ_r as an integral over frequency of its imaginary part, so a purely real $\epsilon_r - 1$ is inconsistent.