Physics 504, Lecture 12
March 3, 2011

1 Multipole Expansion, Vector Spherical Harmonics

Last time we derived the expansion of the Green's function for the scalar helmholtz equation, $(\nabla^2 + k^2)\Psi = -\delta(\vec{x} - \vec{x}\,')$, but we observed that it was awkward to use it for each of the *Cartesian* components of the vector potential $\vec{A}$. Indeed, we found for $\ell = 0$ that $\vec{A}$ is dominated by the electric dipole moment, which looks much like an $\ell = 1$ effect. When we discussed the resonant cavity formed by the Earth and its ionosphere, we considered $\Psi = \vec{r} \cdot \vec{E}$ or $\Psi = \vec{r} \cdot \vec{H}$, which is more compatible with using spherical coordinates. Because away from sources $\vec{E}$ and $\vec{H}$ are *both* divergenceless, each of these Ψ's obeys the free Helmholtz equation away from the sources, and can be expanded as we did in lecture 5: Either

$$\vec{r} \cdot \vec{H}^{(M)}_{\ell m} = \frac{\ell(\ell+1)}{k} g_\ell(kr) Y_{\ell m}(\theta, \phi), \qquad \vec{r} \cdot \vec{E}^{(M)} = 0 \qquad (1)$$

$$\text{or} \quad \vec{r} \cdot \vec{E}^{(E)}_{\ell m} = -Z_0 \frac{\ell(\ell+1)}{k} g_\ell(kr) Y_{\ell m}(\theta, \phi), \qquad \vec{r} \cdot \vec{H}^{(E)} = 0. \qquad (2)$$

for magnetic multipole modes (M) (Eq. 1) or electric multipole modes (E) (Eq. 2). In either case g_ℓ satisfies the spherical Bessel equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) g_\ell(kr) = 0$$

with solutions outside the source region proportional to $h^{(1)}_\ell(kr)$ for outgoing waves. As we found in that lecture, the transverse components are given by

$$\vec{E}^{(M)}_{\ell m} = Z_0 g_\ell(kr) \vec{L} Y_{\ell m}, \qquad \vec{H}^{(M)}_{\ell m} = -\frac{i}{kZ_0} \vec{\nabla} \times \vec{E}^{(M)}_{\ell m} \qquad (3)$$

$$\text{or} \quad \vec{H}^{(E)}_{\ell m} = g_\ell(kr) \vec{L} Y_{\ell m}(\theta, \phi), \qquad \vec{E}^{(E)}_{\ell m} = i\frac{Z_0}{k} \vec{\nabla} \times \vec{H}^{(E)}_{\ell m}. \qquad (4)$$

We define the vector spherical harmonic functions, for $\ell \ge 1$, as

$$\vec{X}_{\ell m}(\theta, \phi) := \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y_{\ell m}(\theta, \phi),$$

with orthogonality properties

$$\begin{aligned}
\int d\Omega \vec{X}^*_{\ell' m'} \vec{X}_{\ell m} &= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega \Big[\frac{1}{2} \left(L^*_+ Y^*_{\ell' m'} \right) (L_+ Y_{\ell m}) \\
&\qquad + \frac{1}{2} \left(L^*_- Y^*_{\ell' m'} \right) (L_- Y_{\ell m}) + \left(L^*_z Y^*_{\ell' m'} \right) (L_z Y_{\ell m}) \Big] \\
&= \int d\Omega \frac{Y^*_{\ell' m'} \left[\frac{1}{2} L_- L_+ + \frac{1}{2} L_+ L_- + L_z^2 \right] Y_{\ell m}}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \\
&= \frac{\sqrt{\ell(\ell+1)}}{\sqrt{\ell'(\ell'+1)}} \int d\Omega\, Y^*_{\ell' m'} Y_{\ell m} \\
&= \delta_{\ell\ell'} \delta_{mm'}
\end{aligned}$$

where $\int d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$, we have used $\vec{L}^{\,2} = \frac{1}{2} L_- L_+ + \frac{1}{2} L_+ L_- + L_z^2$, $\int d\Omega (\vec{L}\Phi)^* \Psi = \int d\Omega \Phi^* \vec{L} \Psi$, and $\int d\Omega Y^*_{\ell' m'} Y_{\ell m} = \delta_{\ell\ell'}\delta_{mm'}$. We also have

$$\begin{aligned}
\int d\Omega \vec{X}^*_{\ell' m'} \cdot \left(\vec{r} \times \vec{X}_{\ell m} \right) &= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega \left(\vec{L}^* Y^*_{\ell' m'} \right) \cdot (\vec{r} \times \vec{L}) Y_{\ell m} \\
&= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega Y^*_{\ell' m'} \vec{L} \cdot (\vec{r} \times \vec{L}) Y_{\ell m} \\
&= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega Y^*_{\ell' m'} \vec{r} \cdot (\vec{L} \times \vec{L}) Y_{\ell m} = 0.
\end{aligned}$$

Note in the last equality, it is not because $\vec{L} \times \vec{L}$ vanishes. Indeed, $\vec{L} \times \vec{L} = \epsilon_{ijk} \hat{e}_i L_j L_k = \frac{1}{2} \epsilon_{ijk} \hat{e}_i [L_j, L_k] = i\vec{L}$ but because $\vec{L}$ is all angular derivatives, it is perpendicular to $\vec{r}$, $\vec{r} \cdot \vec{L} = 0$.

We could reexpress our results for the radiation of arbitrary sources, in terms of the appropriate expansion coefficients a_ℓ multiplying $h^{(1)}_\ell$ for $\vec{r} \cdot \vec{H}$ or $\vec{r} \cdot \vec{E}$. But the justification for claiming $\vec{r} \cdot \vec{E}$ and $\vec{r} \cdot \vec{H}$ satisfy the Helmholtz equation required them to be divergenceless, which $\vec{r} \cdot \vec{E}$ is not in the presence of sources. The trick is to evaluate $\vec{E}' := \vec{E} + i\vec{J}/\omega\epsilon_0$, so $\vec{r} \cdot \vec{E}'$ and $\vec{r} \cdot \vec{H}$ do satisfy inhomogeneous Helmholtz equations with sources given by ρ and $\vec{J}$, with the latter supplemented by any intrinsic magnetization. This is somewhat messy, given in section 9.10, but we will not elaborate. A major use has the sources given by quantum mechanical operators for atomic or nuclear structure, and the vector potential is then a wave function for outgoing photons, giving a decay probabilities rather than radiation power flux. But we will skip this as well, and proceed to discuss scattering of electromagnetic waves.

2 Scattering

Accelerating charges radiate. We have discussed what a specified oscillating current density will do, but we need to also discuss what current density will be created by an incident electromagnetic field. That is, scattering.

If we have some material which will respond to an incident electromagnetic field, it will also radiate field, generally in all directions. Of most interest is an incident plane wave, and the amplitude for an emitted wave in a particular direction. We will consider small scatterers, of size $\ll \lambda$, and an incident plane wave[1]

$$\vec{E}_{\rm inc} = \vec{\epsilon}_i E_i e^{ik\vec{n}_i\cdot\vec{x}}, \qquad \vec{H}_{\rm inc} = \vec{n}_i \times \vec{E}_{\rm inc}/Z_0.$$

where a time dependence $e^{-i\omega t}$ is understood and $k = \omega/c$. If the scatterer is then induced to have electric and magnetic dipole moments $\vec{p}$ and $\vec{m}$ induced by this wave, we will get dipole radiation (for $r \gg \lambda$) as derived earlier

$$\vec{E}_{\rm sc} = \frac{k^2 e^{ikr}}{4\pi\epsilon_0 r}\left[(\hat{r}\times\vec{p})\times\hat{r} - \hat{r}\times\vec{m}/c\right], \qquad \vec{H}_{\rm sc} = \hat{r}\times\vec{E}_{\rm sc}/Z_0.$$

For classical particle dynamics, $\dfrac{d\sigma}{d\Omega}$ is the area of the incident beam which gets scattered into the solid angle $d\Omega$. For a wave, we can't follow individual particles. But the incident energy flux is uniform, and we can define the differential cross section as the power scattered into a given solid angle divided by the incident flux.

The flux is given by

$$\frac{1}{2}\hat{r}\cdot\left(\vec{E}_{\rm sc}\times\vec{H}^*_{\rm sc}\right) = \frac{1}{2Z_0}\hat{r}\cdot\left(\vec{E}_{\rm sc}\times\left(\hat{r}\times\vec{E}^*_{\rm sc}\right)\right) = \frac{1}{2Z_0}\vec{E}_{\rm sc}\cdot\vec{E}^*_{\rm sc},$$

[1] I am using $\vec{\epsilon}_i$ instead of $\vec{\epsilon}_0$ to remove some of the confusion from using the same symbol as for the permittivity of free space ϵ_0. I will also use i (for incident) more generally instead of 0.

as $\hat{r}\cdot\vec{E}_{\rm sc} = 0$.

But the outgoing wave consists of two polarizations, and we can ask what the cross section is for a given polarization, $\vec{\epsilon}$, for an incident wave with polarization $\vec{\epsilon}_i$, with the field projected by $\vec{\epsilon}^*\cdot\vec{E}$. Then we have the polarized differential cross section

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\hat{r},\vec{\epsilon};\hat{n}_i,\vec{\epsilon}_i) &= r^2\frac{\left|\vec{\epsilon}^*\cdot\vec{E}_{\rm sc}\right|^2}{\left|\vec{\epsilon}^*_i\cdot\vec{E}_{\rm inc}\right|^2} \\ &= \frac{k^4}{(4\pi\epsilon_0 E_i)^2}\left|\vec{\epsilon}^*\cdot\vec{p} + (\hat{r}\times\vec{\epsilon}^*)\cdot\vec{m}/c\right|^2.\end{aligned}$$

Because we are assuming the scatterers are small compared to a wavelength, we expect the electric and dipole moment to be quasi-static responses to the applied fields, and thus not dependent on ω, so the overall scattering strength is proportional to $k^4 \propto \omega^4$, which is known as Rayleigh's law.

The responses $\vec{p}$ and $\vec{m}$ can be calculated in simple models. We will consider a uniform dielectric sphere of radius a, with dielectric constant $\epsilon_r = \epsilon/\epsilon_0$, and no magnetization, permeability $\mu = \mu_0$.

2.1 Dielectric Sphere

Last term you found (Jackson 4.56) that a non-magnetic dielectric sphere of radius a has a static induced dipole moment

$$\vec{p} = 4\pi\epsilon_0\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)a^3\vec{E}_{\rm inc},$$

and of course $\vec{m} = 0$. So

$$\frac{d\sigma}{d\Omega}(\hat{r},\vec{\epsilon};\hat{n}_i,\vec{\epsilon}_i) = k^4 a^6\left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2\left|\vec{\epsilon}^*\cdot\vec{\epsilon}_i\right|^2.$$

The scattered wave has the electric field in the plane of the incident polarization and $\hat{r}$; if $\vec{\epsilon}^* \perp \vec{\epsilon}_i$ the amplitude is zero.

If the incident wave is unpolarized, say coming in the z direction, we may take the average over polarization in ϕ, $\vec{\epsilon}_i = (\cos\phi, \sin\phi, 0)$. If we are looking at an angle θ, say with $\hat{r} = (\sin\theta, 0, \cos\theta)$, the two polarization vectors are $\vec{\epsilon}_\parallel = (\cos\theta, 0, \sin\theta)$ in the scattering plane and $\vec{\epsilon}_\perp = (0, 1, 0)$ perpendicular to it.

Then $\left|\vec{\epsilon}_{\parallel}^{\,*}\cdot\vec{\epsilon}_i\right|^2 = \cos^2\theta\cos^2\phi$ and $|\vec{\epsilon}_{\perp}^{\,*}\cdot\vec{\epsilon}_i|^2 = \sin^2\phi$, with average values (over ϕ of $\frac{1}{2}\cos^2\theta$ and $\frac{1}{2}$ respectively. So

$$\begin{aligned}\frac{d\sigma_\parallel}{d\Omega} &= \frac{k^4a^6}{2}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2\cos^2\theta,\\ \frac{d\sigma_\perp}{d\Omega} &= \frac{k^4a^6}{2}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2.\end{aligned}$$

The *polarization* is defined by the difference over the sum,

$$\Pi(\theta) := \frac{\dfrac{d\sigma_\perp}{d\Omega}-\dfrac{d\sigma_\parallel}{d\Omega}}{\dfrac{d\sigma_\perp}{d\Omega}+\dfrac{d\sigma_\parallel}{d\Omega}} \qquad \left(=\frac{\sin^2\theta}{1+\cos^2\theta}\text{ for dielectric sphere}\right)$$

If we don't measure the polarization of the scattered light, the unpolarized cross section is the sum of the two,

$$\frac{d\sigma}{d\Omega} = \frac{k^4a^6}{2}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2\left(1+\cos^2\theta\right)$$

and the total cross section is the integral of this over $d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$,

$$\begin{aligned}\sigma &= \pi k^4a^6\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2\int_{-1}^{1} d(\cos\theta)\left(1+\cos^2\theta\right)\\ &= \frac{8\pi}{3}k^4a^6\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2.\end{aligned}$$

In Jackson §10.1C a small perfectly conducting sphere is considered as the scatterer, but we will skip that.