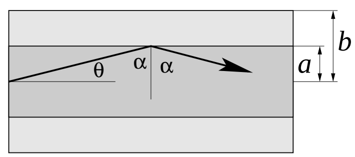
Physics 504, Lecture 10
Feb. 24, 2011

1 Geometrical Fiber Optics

The wave guides considered so far contained their fields within conducting walls, but we know from studying total internal reflection in elementary optics that it is also possible to contain fields by changes in the index of refraction. An extremely important application is fiber optics.

A fiber optic cable is a silica fiber with an index of refraction which varies as a function of radius. The simplest form is a core of radius a and index of refraction n_1, surrounded by a shell of outer radius b and having an index of refraction $n_0 < n_1$. If the angle of incidence $\alpha > \alpha_c = \sin^{-1}(n_0/n_1)$, there will be total internal reflection and the light will be confined. It is more convenient, in discussing optical fibers, to use the angle θ which the light makes with the axis of the fiber, so the condition for total internal reflection is $\theta < \theta_{\rm max} = \cos^{-1}(n_0/n_1)$.



Of course this discussion was in terms of geometrical optics, suitable if the wavelength of the light is negligible compared to the geometrical distances, $\lambda \ll a$. Optical fibers come in two categories, multimode and single mode. For multimode fibers, $a \approx 25\,\mu$m, $b \approx 75\,\mu$m, and the light used is generally near infrared, $\lambda \sim 0.85\,\mu$m, so geometrical optics is a reasonable approach, though we shall see that interference effects are still relevant. Single mode fiber is smaller, $a \approx 2\,\mu$m, and we need to treat these as waveguides.

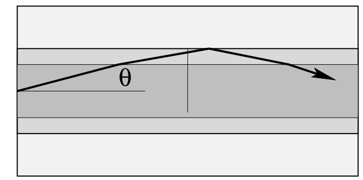
Consider the simple multimode fiber, and define

$$\Delta = \frac{n_1^2 - n_0^2}{2n_1^2} \approx 1 - \frac{n_0}{n_1},$$

which is often about 0.01. As it is small, $\cos\theta_{\rm max} \approx 1 - \frac{1}{2}\theta_{\rm max}^2 = 1 - \Delta$, so $\theta_{\rm max} \approx \sqrt{2\Delta}$. There is an uncertainty principle between the localization of a wave and its wave number, which limits the number of modes that can be transmitted. We may think of this quantum mechanically, where the density of modes is given by $\int (dpdq/2\pi\hbar)^D$ for each mode (in D dimensions). The

coordinate integral is $\int d^2q = \pi a^2$, and as $|\vec{k}_\perp| \leq k_z \tan\theta_{\rm max} = k_z\sqrt{2\Delta}$, $\int d^2p = \hbar^2 \int d^2k = 2\pi\hbar^2 k_z^2 \Delta$. There are 2 polarizations, so the number of modes that can propagate is roughly $N = k_z^2 a^2 \Delta = \frac{1}{2}V^2$, where $V := ka\sqrt{2\Delta}$ is called the *fiber parameter*. For a multimode fiber N is about 100, but for a single mode it is 2, one for each polarization.

There is a problem with this simple arrangement, as the distance light travels to get a distance z down the fiber is $z\sec\theta$, so light with different θ values will travel different optical distances to get to the same point, and will interfere. This can be ameliorated by having more than one transition, or even a continuum. In fact, for next week you will find how it can be fixed with a continuous distribution of $n(r)$.



To analyze such a situation, where $\epsilon(\vec{x})$ varies smoothly, and assuming $\mu = \mu_0$ as the fiber is not magnetic, we may write Maxwell's equations, having fourier transformed time to discuss a particular frequency, as

$$\begin{aligned}
\vec{\nabla}\cdot\epsilon\vec{E} &= 0 = \left(\vec{\nabla}\epsilon\right)\cdot\vec{E} + \epsilon\vec{\nabla}\cdot\vec{E} \\
\vec{\nabla}\times\vec{E} &= -\mu_0\frac{\partial\vec{H}}{\partial t} = i\mu_0\omega\vec{H} \\
\vec{\nabla}\times\vec{H} &= \frac{\partial\epsilon\vec{E}}{\partial t} = -i\omega\epsilon\vec{E} \\
\vec{\nabla}\cdot\vec{H} &= 0.
\end{aligned}$$

So

$$\begin{aligned}
\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) &= -\nabla^2\vec{E} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) = i\mu_0\omega\vec{\nabla}\times\vec{H} = \mu_0\omega^2\epsilon\vec{E} \\
&= -\nabla^2\vec{E} - \vec{\nabla}\left(\frac{1}{\epsilon}\left(\vec{\nabla}\epsilon\right)\cdot\vec{E}\right) \\
\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right) &= -\nabla^2\vec{H} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right) = -i\omega\vec{\nabla}\times\left(\epsilon\vec{E}\right) \\
-\nabla^2\vec{H} &= -i\omega\left(\vec{\nabla}\epsilon\right)\times\vec{E} - i\omega\epsilon\vec{\nabla}\times\vec{E} = -i\omega\left(\vec{\nabla}\epsilon\right)\times\vec{E} + \mu_0\omega^2\epsilon\vec{H}.
\end{aligned}$$

or

$$\begin{aligned}
\nabla^2\vec{E} + \mu_0\omega^2\epsilon\vec{E} + \vec{\nabla}\left(\frac{1}{\epsilon}\left(\vec{\nabla}\epsilon\right)\cdot\vec{E}\right) &= 0 \\
\nabla^2\vec{H} + \mu_0\omega^2\epsilon\vec{H} - i\omega\left(\vec{\nabla}\epsilon\right)\times\vec{E} &= 0
\end{aligned}$$

We can simplify these equations if we assume that ϵ varies slowly compared to a wavelength, so

$$\nabla\epsilon \ll \frac{\epsilon}{\lambda} = \frac{\epsilon\omega}{c}.$$

The other terms in the equations are of order ω^2/c^2 times E or H respectively, ($\epsilon E \sim H/c$) so the $\nabla\epsilon$ terms may be dropped as small. Then the components of both $\vec{E}$ and $\vec{H}$ satisfy[1]

$$\left(\nabla^2 + \frac{\omega^2}{c^2}n^2(\vec{r})\right)\psi(\vec{r}) = 0. \tag{1}$$

This equation, which describes the rapidly oscillating function ψ, can be replaced by a more gradually varying function by using the Eikonal, writing

$$\psi(\vec{r}) = e^{i\omega S(\vec{r})/c}$$

$$\text{so}\quad \nabla^2\psi = \vec{\nabla}\cdot\left(\frac{i\omega}{c}\vec{\nabla}S e^{i\omega S(\vec{r})/c}\right) = \left[\frac{i\omega}{c}\nabla^2 S - i\left(\frac{\omega}{c}\right)^2\left(\vec{\nabla}S\right)^2\right]e^{i\omega S(\vec{r})/c}.$$

This is $-(\omega^2 n^2/c^2)e^{i\omega S/c}$ from (1), so

$$n^2(\vec{r}) - \vec{\nabla}S\cdot\vec{\nabla}S = -i\frac{c}{\omega}\nabla^2 S.$$

$c/\omega \sim \lambda$, so as ∇S varies on the same scale as $n(\vec{r})$, which is slowly compared to $1/\lambda$, the right hand side can be dropped, and we have the *eikonal approximation*

$$\vec{\nabla}S\cdot\vec{\nabla}S = n^2(\vec{r}). \tag{2}$$

This equation doesn't tell us the direction in which S changes but it does tell us that the rate is just $n(\vec{r})$, so in following a particular ray's path we can define a unit vector $\hat{k}(\vec{r})$ such that $\vec{\nabla}S = n(\vec{r})\hat{k}(\vec{r})$. Near a point r_0 we may expand

$$\psi(\vec{r}) = e^{i\omega\left(S(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}S\right)/c} = e^{i\omega S(\vec{r}_0)/c}\,e^{i\omega\hat{k}\cdot(\vec{r}-\vec{r}_0)n(\vec{r})/c},$$

[1] Jackson uses $\vec{x}$ instead of $\vec{r}$ in most of these equations, to emphasize that n depends only on the transverse location, and not on the axial coordinate z. But clearly the wave function ψ and the eikonal S do depend on z along a given ray.

so it is locally a plane wave with $|\vec{k}| = \omega n(\vec{r})/c$, suitable for a wave with speed $c/n(\vec{r})$ as one might expect. If s is the distance measured along the path of the ray, $d\vec{r}/ds = \hat{k}$, $n(\vec{r})d\vec{r}/ds = \vec{\nabla}S$, so

$$\frac{d}{ds}\left(n(\vec{r})\frac{d\vec{r}}{ds}\right) = \frac{d}{ds}\vec{\nabla}S = \vec{\nabla}\left.\frac{dS}{ds}\right|_\Gamma = \vec{\nabla}n(\vec{r}). \tag{3}$$

(In the penultimate expression Γ represents the ray's path.)

In general, a ray will be directed in a direction with a large axial component, a radial component, and an aximuthal component. If the latter is zero, the ray will pass through the axis, and the ray is called *meridional*. Rays with a nonzero azimuthal component will never pass directly through the axis, and are called *skew rays*. In terms of wave functions, such rays correspond to azimuthal 'quantum' numbers $m \neq 0$, and have vanishing intensity at $\rho = 0$ (on the axis). Following Jackson we will avoid complication and discuss only meridional rays, which is also equivalent (for geometric optics, at least) to discussing a plane slab. So the ρ direction will be renamed x, and we consider a ray confined to the xz plane, where z is the overall direction of motion. We assume n doesn't depend on z.

The x and z components of Eq (3) are

$$\frac{d}{ds}\left(n(x)\sin\theta\right) = \frac{dn(x)}{dx}, \qquad \frac{d}{ds}\left(n(x)\cos\theta\right) = \frac{dn(\vec{r})}{dz} = 0.$$

Thus $n(x)\cos\theta$ is a constant along the path, and if $\theta < \theta_{\rm max}$ the path will turn around at $x_{\rm max}$, at which $\cos\theta = 1$, so the constant value of $n(x)\cos\theta$ is just $\bar{n} := n(x_{\rm max})$, but it is also, of course, $n(0)\cos\theta(0)$.

Note that $dz/ds = \cos\theta = \bar{n}/n$, so the path is determined by rewriting the x component of Eq. (3) as

$$\frac{\bar{n}}{n(x)}\frac{d}{dz}\left(n(x)\frac{\bar{n}}{n(x)}\frac{dx}{dz}\right) = \frac{\bar{n}}{n(x)}\frac{d}{dz}\left(\bar{n}\frac{dx}{dz}\right) = \frac{dn}{dx}$$

$$\text{so}\qquad \bar{n}^2\frac{d^2x}{dz^2} = n(x)\frac{dn(x)}{dx} = \frac{1}{2}\frac{d}{dx}n^2(x).$$

Whether by inspiration from mechanics, viewing $-\frac{1}{2}n^2(x)$ as a potential, or otherwise, if we multiply this equation by $x' := dx/dz$ we get

$$\frac{1}{2}\bar{n}^2\frac{d}{dz}\left(\frac{dx}{dz}\right)^2 = \frac{1}{2}\frac{d}{dz}n^2(x) \Longrightarrow \bar{n}^2 x'^2 = n^2(x) - \bar{n}^2,$$

where the constant of integration $\bar{n}^2$ is determined by $x' = 0$ at $x_{\rm max}$, $n(x_{\rm max}) = \bar{n}$. So the axial distance traveled from the point the ray crossed the axis to reaching radius x is

$$z(x) = \int_0^x \frac{dz}{dx} dx = \bar{n} \int_0^x \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}}.$$

The path is important because if different rays have different path lengths to get to the same (large) displacement z down the fiber, they will interfere. From one crossing of the axis to the next, the ray moves in z a distance

$$Z = 2\bar{n} \int_0^{x{\rm max}} \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}}.$$

The distance travelled is not important, but the optical distance, $\int n(x) ds$ is, because that determines the change in phase, and the optical distance corresponding to the axial displacement Z is

$$\begin{aligned} L_{\rm opt} &= 2\int_0^{x{\rm max}} n(x) \frac{ds}{dz}\frac{dz}{dx}\, dx = 2\int_0^{x{\rm max}} n(x) \frac{n(x)}{\bar{n}} \frac{\bar{n}}{\sqrt{n^2(x) - \bar{n}^2}}\, dx \\ &= 2\int_0^{x{\rm max}} \frac{n^2(x)}{\sqrt{n^2(x) - \bar{n}^2}}\, dx. \end{aligned}$$

The phase difference in a single half-period is not likely to be important, but when the rays travel a large distance z, the total optical path will be $L_{\rm opt} z/Z$. Thus it is ideal if $L_{\rm opt}/Z$ is independent of $\bar{n}$, that is, it is the same for all rays. In problem 8.14 you will show how to accomplish that.

In addition to geometrical dispersion coming from not having $L_{\rm opt}/Z$ independent of $\bar{n}$, there can also be frequency dispersion due to the variation of $n(\vec{x})$ with frequency. This will not effect a very narrow bandwidth signal (one with a very narrow range of ω), but the rate at which information can be carried is proportional to the bandwidth, so we would like to have little dispersion in frequency as well. We also want minimal absorption. These two issues for silica encourage using $\lambda \sim 1.4\mu$m.

We will skip Jackson §8.11.

2 Sources of Waves in Wave Guides

We now turn our attention to the sources of waves in waveguides. With given sources (*i.e.* ignoring back reactions) the equations are still linear (inhomogeneous) in the fields and time-independent, so we can assume everything has a $e^{-i\omega t}$ time dependence. The normal modes of free waves in a waveguide form a complete set of states for the source-free solutions to Maxwell's equations in the guide, though we need to include all the modes, including those whose cutoff frequency is above ω. Far from the sources, however, only the real values of k, for modes with $\omega_\lambda < \omega$, will have appreciable amplitude.

We will expand our fields in the normal modes, which will be indexed by a composite index λ, which includes information about whether the mode is TE or TM (or of other nature, *e.g.* TEM), and also the indices which specify which mode (roughly the angular and radial 'quantum' numbers) For a given mode we have two values of k, either positive (right moving) and negative (left moving), or $\pm i|k|$ for cutoff modes. We will lump the $+i|k|$ modes in with the positive k ones, and write the (+) part of the λ mode as

$$\begin{aligned} \vec{E}^+_\lambda(x,y,z) &= \left[\vec{E}_\lambda(x,y) + \hat{z} E_{z\,\lambda}(x,y)\right] e^{ik_\lambda z} \\ \vec{H}^+_\lambda(x,y,z) &= \left[\vec{H}_\lambda(x,y) + \hat{z} H_{z\,\lambda}(x,y)\right] e^{ik_\lambda z} \end{aligned}$$

where $\vec{E}$ and $\vec{H}$ are purely transverse, and were given in terms of E_z or H_z for TM or TE modes respectively earlier. There are also modes travelling in the negative z direction. The equations are symmetric under $z \leftrightarrow -z$, under which the transverse $\vec{E}_\lambda(x,y)$ is unchanged but $E_{z\,\lambda}(x,y)$ changes sign. The magnetic field is a *pseudo*-vector, so for it the transverse $\vec{H}_\lambda(x,y)$ changes sign but $H_{z\,\lambda}(x,y)$ is unchanged. Thus

$$\begin{aligned} \vec{E}^-_\lambda(x,y,z) &= \left[\vec{E}_\lambda(x,y) - \hat{z} E_{z\,\lambda}(x,y)\right] e^{-ik_\lambda z} \\ \vec{H}^-_\lambda(x,y,z) &= \left[-\vec{H}_\lambda(x,y) + \hat{z} H_{z\,\lambda}(x,y)\right] e^{-ik_\lambda z} \end{aligned}$$

The transverse fields $\vec{E}_\lambda$ form a basis which we can choose to be real and normalized,

$$\int_A \vec{E}_\lambda \cdot \vec{E}_\mu = \delta_{\lambda\mu}.$$

That this can be done, and what this normalization requires for the basis of normal modes for E_z and H_z is elaborated in the slides. With this normalization, and from $\vec{H}_\lambda = Z_\lambda^{-1}\hat{z}\times\vec{E}_\lambda$ we also have

$$\int_A \vec{H}_\lambda\cdot\vec{H}_\mu = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu},$$

and in the time-averaged power expression $\langle P\rangle = \frac{1}{2}\int_A\left(\vec{E}\times\vec{H}\right)\cdot\hat{z}$ we have

$$\int_A\left(\vec{E}_\lambda\times\vec{H}_\mu\right)\cdot\hat{z} = \frac{1}{Z_\lambda}\delta_{\lambda\mu}$$

The above normalization for the $\vec{E}_\lambda$ comes from requiring that the z components satisfy orthogonality conditions and normalization conditions adjusted appropriately. For TM waves, $\vec{E}_\lambda = ik_\lambda\gamma_\lambda^{-2}\vec{\nabla}E_{z\,\lambda}$, so

$$\begin{aligned}\delta_{\lambda\mu} &= \int_A \vec{E}_\lambda\cdot\vec{E}_\mu = -\frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A\vec{\nabla}E_{z\,\lambda}\cdot\vec{\nabla}E_{z\,\mu} = \frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A E_{z\,\lambda}\nabla^2E_{z\,\mu}\\ &= -\frac{k_\lambda k_\mu}{\gamma_\lambda^2}\int_A E_{z\,\lambda}E_{z\,\mu},\end{aligned}$$

where in the integration by parts (third =) the surface term vanishes as $E|_S = 0$, and for the fourth $\nabla^2E_{z\,\mu} = -\gamma_\mu^2E_{z\,\mu}$, so

$$\int_A E_{z\,\lambda}E_{z\,\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu},$$

while for TE waves the same argument for H gives

$$\int_A H_{z\,\lambda}H_{z\,\mu} = -\frac{\gamma_\lambda^2}{Z_\lambda^2k_\lambda^2}\delta_{\lambda\mu}.$$

For a rectangular waveguide, $0\le x\le a\times 0\le y\le b$, the modes are labelled by integers m and n, with

TM waves: $\psi|_S = 0$

$$\begin{aligned}E_{z\,mn} &= \psi = \frac{-2i\gamma_{mn}}{k_\lambda\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\ E_{x\,mn} &= \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\ E_{y\,mn} &= \frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\end{aligned}$$

TE waves: $\left.\frac{\partial\psi}{\partial n}\right|_S = 0$

$$\begin{aligned}H_{z\,mn} &= \psi = \frac{-2i\gamma_{mn}}{k_\lambda Z_\lambda\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\\ E_{x\,mn} &= \frac{-2\pi n}{\gamma_{mn}b\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\ E_{y\,mn} &= \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\end{aligned}$$

where

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right).$$

The functional form of ψ is immediately apparent from the boundary conditions, and for TM modes $\vec{E} = ik\vec{\nabla}\psi/\gamma^2$, and for TE modes $\vec{E} = -iZk\hat{z}\times\vec{\nabla}\psi/\gamma^2$. The overall constants are determined from the normalization $\int_A E_x^2+E_y^2 = 1$, except that for TE modes, we need an extra factor of $1/\sqrt{2}$ for each n or m which is zero, as $\int\cos^2(m\pi x/a) = a(1+\delta_{m0})/2$.

2.1 Expansion of free waves

In our waveguide, an arbitrary field configuration in the absence of sources can be described by expanding in normal modes with positive and negative (or $+i$ and $-i$) wave numbers, as described above. Thus a total field

$$\vec{E} = \vec{E}^+ + \vec{E}^-,\qquad \vec{H} = \vec{H}^+ + \vec{H}^-,$$

with

$$E^{\pm} = \sum_\lambda A^{\pm}_\lambda \vec{E}^{\pm}_\lambda, \qquad H^{\pm} = \sum_\lambda A^{\pm}_\lambda \vec{H}^{\pm}_\lambda,$$

can describe an arbitrary field configuration in a region that has no sources, with the A's constant (independent of z) in such a section of the wave guide. The coefficients are determined by the transverse components $\vec{E}$ and $\vec{H}$ along any cross section, for $\vec{E}$ has expansion coefficients $A^+_\lambda e^{ik_\lambda z} + A^-_\lambda e^{-ik_\lambda z}$ while $\vec{H}$ has expansion coefficients $A^+_\lambda e^{ik_\lambda z} - A^-_\lambda e^{-ik_\lambda z}$. This gives the expansion coefficients (taking $z = 0$) as

$$A^{\pm}_\lambda = \frac{1}{2}\int_A \vec{E}\cdot\vec{E}_\lambda \pm Z^2_\lambda \vec{H}\cdot\vec{H}_\lambda.$$

2.2 Localized Sources

We have described the waves that can propagate in the waveguide, but what actually produces such waves? We will consider a localized source with current density $\vec{J}(\vec{x})e^{-i\omega t}$ confined to some region $z \in [z_-, z_+]$, at the ends of which we imagine cross sections denoted by S_-, S_+. We will assume there are no sources or obstacles to the right of S_+ or to the left of S_-, so at S_+ there is no amplitude for any mode with negative k or with $-i|k|$, which would represent left-moving waves or exponential blow up (as $z \to +\infty$). The reverse is true at S_-, so

$$\vec{E} = \sum_{\lambda'} A^+_{\lambda'}\vec{E}^+_{\lambda'}, \qquad \vec{H} = \sum_{\lambda'} A^+_{\lambda'}\vec{H}^+_{\lambda'} \qquad \text{at } S_+$$

$$\vec{E} = \sum_{\lambda'} A^-_{\lambda'}\vec{E}^-_{\lambda'}, \qquad \vec{H} = \sum_{\lambda'} A^-_{\lambda'}\vec{H}^-_{\lambda'} \qquad \text{at } S_-$$

In between, we have the full Maxwell equations (with sources),

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\mu_0\vec{H}, \quad \vec{\nabla}\times\vec{H} = \vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{J} - i\omega\epsilon_0\vec{E},$$

while the normal modes obey Maxwell equations without sources:

$$\vec{\nabla}\times\vec{H}^{\pm}_\lambda = -i\omega\epsilon_0\vec{E}^{\pm}_\lambda, \quad \vec{\nabla}\times\vec{E}^{\pm}_\lambda = i\omega\mu_0\vec{H}^{\pm}_\lambda.$$

If we apply the identity

$$\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right) = \left(\vec{\nabla}\times\vec{A}\right)\cdot\vec{B} - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right),$$

we find

$$\begin{aligned}
&\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^{\pm}_\lambda - \vec{E}^{\pm}_\lambda\times\vec{H}\right)\\
&\quad = \left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{H}^{\pm}_\lambda - \vec{E}\cdot\left(\vec{\nabla}\times\vec{H}^{\pm}_\lambda\right) - \left(\vec{\nabla}\times\vec{E}^{\pm}_\lambda\right)\cdot\vec{H} + \vec{E}^{\pm}_\lambda\cdot\left(\vec{\nabla}\times\vec{H}\right)\\
&\quad = i\omega\mu_0\vec{H}\cdot\vec{H}^{\pm}_\lambda + i\omega\epsilon_0\vec{E}\cdot\vec{E}^{\pm}_\lambda - i\omega\mu_0\vec{H}^{\pm}_\lambda\cdot\vec{H} + \vec{E}^{\pm}_\lambda\cdot\left(\vec{J} - i\omega\epsilon_0\vec{E}\right)\\
&\quad = \vec{J}\cdot\vec{E}^{\pm}_\lambda
\end{aligned}$$

If we integrate this over the volume between S_- and S_+, using Gauss' theorem and the boundary condition that $\vec{E}_\parallel = 0$ at the conductor's surface,

$$\int_S \left(\vec{E}\times\vec{H}^{\pm}_\lambda - \vec{E}^{\pm}_\lambda\times\vec{H}\right)\cdot\hat{n} = \int_V \vec{J}\cdot\vec{E}^{\pm}_\lambda,$$

where S consists of S_+ with $\hat{n} = \hat{z}$, and S_- with $\hat{n} = -\hat{z}$.

Let's take the upper sign. The contribution from S_+ is can be reduced to an integral over A at $z = 0$:

$$\begin{aligned}
&\sum_{\lambda'} A^+_{\lambda'}\int_{S_+}\left(\vec{E}^+_{\lambda'}\times\vec{H}^+_\lambda - \vec{E}^+_\lambda\times\vec{H}^+_{\lambda'}\right)_z\\
&\quad = \sum_{\lambda'} A^+_{\lambda'}\int_{S_+}\left(\vec{E}_{\lambda'}\times\vec{H}_\lambda - \vec{E}_\lambda\times\vec{H}_{\lambda'}\right)_z e^{i(k_\lambda+k_{\lambda'})z}\\
&\quad = \sum_{\lambda'} A^+_{\lambda'}\int_A\left(\vec{E}_{\lambda'}\times\left(Z^{-1}_\lambda\hat{z}\times\vec{E}_\lambda\right) - \vec{E}_\lambda\times\left(Z^{-1}_{\lambda'}\hat{z}\times\vec{E}_{\lambda'}\right)\right)_z e^{i(k_\lambda+k_{\lambda'})z}\\
&\quad = \sum_{\lambda'} A^+_{\lambda'}\int_A\left(\frac{1}{Z_\lambda}\vec{E}_{\lambda'}\cdot\vec{E}_\lambda - \frac{1}{Z_{\lambda'}}\vec{E}_\lambda\cdot\vec{E}_{\lambda'}\right)e^{i(k_\lambda+k_{\lambda'})z}\\
&\quad = \sum_{\lambda'} A^+_{\lambda'}\delta_{\lambda\lambda'}\left(\frac{1}{Z_\lambda} - \frac{1}{Z'_\lambda}\right)e^{i(k_\lambda+k_{\lambda'})z} = 0.
\end{aligned}$$

On the other hand, the contribution from S_- is

$$\begin{aligned}
&\sum_{\lambda'} A^-_{\lambda'}\int_{S_-} -\left(\vec{E}^-_{\lambda'}\times\vec{H}^+_\lambda - \vec{E}^+_\lambda\times\vec{H}^-_{\lambda'}\right)\cdot\hat{z}\\
&\quad = \sum_{\lambda'} A^-_{\lambda'}\int_{S_-} -\left(\vec{E}_{\lambda'}\times\vec{H}_\lambda + \vec{E}_\lambda\times\vec{H}_{\lambda'}\right)\cdot\hat{z}\, e^{i(k_\lambda-k_{\lambda'})z}\\
&\quad = -\sum_{\lambda'} A^-_{\lambda'}\frac{2}{Z_\lambda}\delta_{\lambda\lambda'} = -\frac{2}{Z_\lambda}A^-_\lambda
\end{aligned}$$

so

$$A^-_\lambda = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}^+_\lambda.$$

The same argument for the lower sign, as spelled out in the book, gives the equation with the superscript signs reversed, so both are

$$A_\lambda^{\pm} = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}_\lambda^{\mp}.$$

In addition to sources due to currents, we may have contributions due to obstacles or holes in the conducting boundaries. These can be treated as additional surface terms in Gauss' law (by excluding obstacles from the region of integration V), but this requires knowing the full fields at the surface of the obstacles or the missing parts of the waveguide conductor. This is treated in §9.5B, but we won't discuss it here.

So finally we are at the end of Chapter 8.