Physics 504, Lecture 19
April 8, 2010

1 Charged Paricle Energy Loss

We now turn to the motion of relativistic charged particles moving through media, interacting on an atomic scale. One thing that will happen is that they will scatter with electrons. If the energy transfer is large enough, we may be able to ignore the fact that the electrons had been bound in atomic states, and treat it as having been a free electron at rest, as we always do when discussing the Compton effect for an incident X-ray.

Let us take the speed, mass and charge of our swiftly moving particle to be v, M and ze, with $E = M\gamma c^2$, $P = M\beta\gamma c$. The electron has mass m and charge $-e$.

Unless the swift particle is an electron or a positron, it will be far heavier than the electron, $M >> m$, and we can treat the scattering as Coulomb scattering of the electron in the rest frame of the swift particle. Then we have Rutherford scattering. The electron will scatter with a distribution of angles given by the Rutherford scattering formula, which Jackson claims in well-known:

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2vp}\right)^2 \frac{1}{\sin^4(\theta/2)}, \qquad (1)$$

where $p = m\beta\gamma c$ is the momentum of the electron. Jackson knows well a lot of things the rest of us don't know at all, for although the non-relativistic formula, with $p \to mv$, is indeed well known, the relativistic version surely is not. The nonrelativistic expression is easily derivable from conservation of energy, angular momentum, and the Runge-Lenz vector, so that the relation between impact parameter b and scattering angle θ can be derived simply by looking at the conserved values infinitely before and after the collision. Unfortunately the Runge-Lenz vector is not conserved relativistically, though people[1] have discussed a rotating version of it. Jackson's formula is derivable[2].

[1] Yoshida, Phys. Rev. A38 (88) 19.

[2] See the Appendix of J. Huschilt and W. E. Baylis, Phys. Rev. D17 (78) 985.

In the "laboratory", *i.e.* the rest frame of the medium, we are not interested in the scattering angle of the electron but are interested in the change of momentum of the swift particle, so we can reexpress this cross section in terms of the 4-momentum transfer, $Q^2 = -(p'^\mu - p^\mu)^2 > 0$, which for elastic scattering will be

$$Q^2 = 4p^2 \sin^2(\theta/2), \quad dQ^2 = 2p^2 \sin\theta\, d\theta,$$

so

$$d\Omega = 2\pi \sin\theta\, d\theta = \frac{\pi}{p^2} dQ^2, \qquad \frac{d\sigma}{dQ^2} = \frac{\pi}{p^2}\frac{d\sigma}{d\Omega} = 4\pi\left(\frac{ze^2}{vQ^2}\right)^2.$$

We don't need to worry about Lorentz transforming $d\sigma$ as that is the area transverse to the boost back to the lab frame. In the swift frame $P^\mu = (Mc, \vec{0})$, $p^\mu = (mc\gamma, -m\gamma\vec{v})$, so $P\cdot p = Mmc^2\gamma$, so $\beta^2 = (Mmc^2/P\cdot p)^2$. What is of most interest is the energy lot to the electron, $T = (p'^0 - p^0)c$ in the lab, where $p^\mu = (mc, \vec{0})$, so $mT = p\cdot(p' - p) = p\cdot p' - p^2 = -\frac{1}{2}(p' - p)^2 = \frac{1}{2}Q^2$. Replacing Q^2 by $2mT$ on both sides of the cross section equation,

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 e^4}{mv^2 T^2}.$$

We will use this formula to find the rate of energy loss as the swift particle penetrates the medium, but first we must note the limits on its applicability. It seems to suggest a cross section for loosing arbitrary amounts of energy, but as

$$T = \frac{Q^2}{2m} = 2\frac{p^2}{m}\sin^2\left(\frac{\theta}{2}\right) \le 2m(c\beta\gamma)^2,$$

there is an upper limit on T. There is also a lower limit on the use of the formula for $d\sigma/dT$, for unless enough energy is transferred to the electron to free it from the confines of the atom, quantum mechanics will restrict its ability to absorb energy from the scattering. Jackson describes this lower limit on T as $\hbar\langle\omega\rangle$, an effective binding energy, and also as ϵ. He gives some proviso's about corrections due to the spin of the electron and correction necessary if the incoming speed is so great that the electron has a momentum in the swift particle's rest frame that is sufficient to disrupt it, even though $m << M$. This can be estimated to be when $m\gamma \sim M$, or $E = Mc^2\gamma \sim c^2M^2/m$.

Putting aside these objections, we may ask at what rate (in x, the penetration distance) our swift particle looses energy when passing though a material with N atoms per unit volume and Z electrons per atom. It looses energy $T \pm dT/2$ for each of $ZN dx d\sigma/dT$ electrons it scatters off of, so

$$\begin{aligned} \frac{dE}{dx} &= NZ\int_\epsilon^{T\max} T\frac{d\sigma}{dT}\,dT = 2\pi NZ\frac{z^2e^4}{mv^2}\int_\epsilon^{T\max}\frac{1}{T}\,dT \\ &= 2\pi NZ\frac{z^2e^4}{mv^2}\ln\left(\frac{T\max}{\epsilon}\right) \\ &= 2\pi NZ\frac{z^2e^4}{mv^2}\ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right). \end{aligned}$$

There are lots of corrections to this formula. Dirac spin of the electron subtracts β^2 from the logarithm. Energy losses from scattering of less than ϵ cannot be neglected. The corrections these make to dE/dx doubles the coefficient of the $\ln\gamma$ and adds a term to it which doesn't grow much. So these leave unchanged the basic features that

- For low velocities the energy loss is inversely proportional to β^2. As the coeffient's dependence on the material is NZ which is roughly proportional to N times the atomic mass, and hence to the density of the material, the energy loss per gram per cm^2 is approximately the same for most materials.
- For high velocities, β saturates near 1, but γ can grow, so the energy loss per gram/cm^2 grows logarithmically with γ or energy. Therefore there is a minimum ionizing value, which is somewhere around $\beta\gamma = 3$.

2 Coherent Effects

In the previous discussion we assumed the charged particle interacted with the individual electrons incoherently, which is a dubious approximation for impact parameters on the scale of atomic distances. Part of the scattering mechanism is more correctly thought of in terms of the polarization the charged particle induces in the medium as it passes through. Let us write Maxwell's equations in a medium we assume to have a uniform polarizability $\epsilon(\omega)$ but no magnetization, in our new (Gaussian) units,

$$\vec\nabla\cdot\vec D = 4\pi\rho \tag{2}$$

$$\vec\nabla\cdot\vec B = 0 \tag{3}$$

$$\vec\nabla\times\vec B - \frac{1}{c}\frac{\partial\vec D}{\partial t} = \frac{4\pi}{c}\vec J \tag{4}$$

$$\vec\nabla\times\vec E + \frac{1}{c}\frac{\partial\vec B}{\partial t} = 0 \tag{5}$$

with the vector and scalar potentials giving

$$\vec B = \vec\nabla\times\vec A, \qquad \vec E = -\vec\nabla\Phi - \frac{1}{c}\frac{\partial\vec A}{\partial t}.$$

If we write all of these as Fourier transforms,

$$F(\vec x, t) = \frac{1}{(2\pi)^2}\int d^3k\int d\omega F(\vec k,\omega)e^{i\vec k\cdot\vec x - i\omega t},$$

we get

$$\begin{aligned} \vec E(\vec k,\omega) &= -i\vec k\Phi(\vec k,\omega) + \frac{i\omega}{c}\vec A(\vec k,\omega) \\ \vec D(\vec k,\omega) &= -i\epsilon(\omega)\vec k\,\Phi(\vec k,\omega) + \frac{i\omega\epsilon(\omega)}{c}\vec A(\vec k,\omega) \\ \vec B(\vec k,\omega) &= i\vec k\times\vec A(\vec k,\omega) \end{aligned}$$

so (2) and (4) become

$$\begin{aligned} \epsilon(\omega)k^2\Phi(\vec k,\omega) - \frac{\omega\epsilon(\omega)}{c}\vec k\cdot\vec A(\vec k,\omega) &= 4\pi\rho(\vec k,\omega) \\ -\vec k\times\left(\vec k\times\vec A(\vec k,\omega)\right) + \frac{\omega}{c}\epsilon(\omega)\vec k\,\Phi(\vec k,\omega) - \frac{\omega^2\epsilon(\omega)}{c^2}\vec A(\vec k,\omega) &= \frac{4\pi}{c}\vec J(\vec k,\omega) \end{aligned}$$

The gauge invariance $A^\mu \to A^\mu - \partial^\mu\Lambda$, or $\vec A \to \vec A + \vec\nabla\Lambda$, $\Phi \to \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$, applies even in macroscopic media, so we may require a modified Lorenz gauge condition

$$\frac{\epsilon}{c}\frac{\partial\Phi}{\partial t} + \vec\nabla\cdot\vec A = 0, \quad \text{or } \vec k\cdot\vec A(\vec k,\omega) = \frac{\omega\epsilon(\omega)}{c}\Phi.$$

Then we can write our equations as

$$\begin{aligned} \epsilon(\omega)k^2\Phi(\vec k,\omega) - \frac{\omega^2\epsilon^2(\omega)}{c^2}\Phi(\vec k,\omega) &= 4\pi\rho(\vec k,\omega) \\ k^2\vec A(\vec k,\omega) - \frac{\omega^2\epsilon(\omega)}{c^2}\vec A(\vec k,\omega) &= \frac{4\pi}{c}\vec J(\vec k,\omega) \end{aligned}$$

We are going to consider the field set up by our swift particle which we can consider moving at a nearly constant velocity $\vec{v}$. We will analyze the effect this has on the electrons in atoms a distance b away from that path, to find the energy lost to such electrons. We are therefore assuming the loss of energy is slow enough to ignore the change in $\vec{v}$ while calculating the effect on the local atoms.

Thus the source of the field is the free charge and current

$$\rho(\vec{x},t) = ze\delta^3(\vec{x}-\vec{v}t), \qquad \vec{J}((\vec{x},t) = \vec{v}\rho(\vec{x},t) = ze\vec{v}\delta^3(\vec{x}-\vec{v}t),$$

which means the fourier transformed source is

$$\begin{aligned}\rho(\vec{k},\omega) &= \frac{ze}{(2\pi)^2}\int d^3x dt\, \delta^3(\vec{x}-\vec{v}t)e^{-i\vec{k}\cdot\vec{x}+i\omega t} = \frac{ze}{(2\pi)^2}\int dt\, e^{-i(\vec{k}\cdot\vec{v}-\omega)t} \\ &= \frac{ze}{2\pi}\delta(\omega-\vec{k}\cdot\vec{v})\end{aligned}$$

and $\vec{J}(\vec{k},\omega) = \vec{v}\rho(\vec{k},\omega)$. In Fourier space the equations for Φ and $\vec{A}$ become trivial to solve:

$$\begin{aligned}\Phi(\vec{k},\omega) &= \frac{2ze}{\epsilon(\omega)}\frac{\delta(\omega-\vec{k}\cdot\vec{v})}{k^2-\omega^2\epsilon(\omega)/c^2} \\ \vec{A}(\vec{k},\omega) &= \frac{\vec{v}\epsilon(\omega)}{c}\Phi(\vec{k},\omega) \\ \vec{E}(\vec{k},\omega) &= -i\vec{k}\Phi(\vec{k},\omega) + i\frac{\omega}{c}\vec{A}(\vec{k},\omega) = \left(-i\vec{k}+i\frac{\omega\epsilon(\omega)}{c^2}\vec{v}\right)\Phi(\vec{k},\omega).\end{aligned}$$

To evaluate the effect of this field on electrons in atoms we make use of the model described in §7.5, in which the electrons in an atom are considered harmonic oscillators with natural frequencies ω_j, damping constants γ_j and oscillator strengths f_j, (with $\sum f_j = Z$), which respond to a local electric field $\vec{E}(\omega)$ with

$$\vec{x}_j(\omega) = -\frac{e}{m}\frac{\vec{E}(\omega)}{\omega_j^2-\omega^2-i\omega\gamma_j}.$$

In terms of this model the dielectric constant is

$$\epsilon(\omega) = 1 + \frac{4\pi Ne^2}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}.$$

Each of these electrons will absorb an energy

$$\begin{aligned}\Delta E &= -e\int_{-\infty}^{\infty} dt\, \vec{v}_j(t)\cdot\vec{E}(\vec{x},t) \\ &= -\frac{e}{2\pi}\int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} d\omega\left(-i\omega x_j(\omega)e^{-i\omega t}\right)\int_{-\infty}^{\infty} d\omega'\vec{E}^*(\omega')e^{i\omega' t} \\ &= ie\int_{-\infty}^{\infty} d\omega\,\omega x_j(\omega)\vec{E}^*(\omega) = 2e\ \mathrm{Re}\ \int_0^{\infty} d\omega\, i\omega x_j(\omega)\vec{E}^*(\omega)\end{aligned}$$

where in the last line we made use of the requirement, because $\vec{x}(t)$ and $\vec{E}(t)$ are real, that $\vec{x}(-\omega) = \vec{x}^*(\omega)$ and the same for $\vec{E}$. Let us take the velocity of the swift particle along the x axis and evaluate this loss of energy for an atom at location $(0,b,0)$, so the required electric field is

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\vec{E}(\vec{k},\omega)e^{ik_2b}.$$

Then the loss of energy from an atom a distance b away is

$$-\Delta E = \frac{2e^2}{m}\sum_j f_j\ \mathrm{Re}\ \int_0^{\infty} d\omega\, \frac{i\omega|\vec{E}|^2}{\omega_j^2-\omega^2-i\omega\gamma_j},$$

and as there are $2\pi Nbdb$ such atoms per unit distance along the particle's path, the energy loss per unit distance is

$$\begin{aligned}\frac{dE}{dx} &= \int_0^{\infty} b\,db\ \mathrm{Re}\ \int_0^{\infty} d\omega\, i\omega|\vec{E}|^2\frac{4\pi Ne^2}{m}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j} \\ &= \int_0^{\infty} b\,db\ \mathrm{Re}\ \int_0^{\infty} d\omega\, i\omega|\vec{E}|^2\left(\epsilon(\omega)-1\right).\end{aligned}$$

We will need to evaluate $|\vec{E}|^2$. This is a bit of a slog. In Fourier space we have

$$\vec{E}(\vec{k},\omega) = \left(-i\vec{k}+i\frac{\omega\epsilon(\omega)}{c^2}\vec{v}\right)\frac{2ze}{\epsilon(\omega)}\frac{\delta(\omega-k_1v)}{k^2-\omega^2\epsilon(\omega)/c^2},$$

so

$$\begin{aligned}\vec{E}(\vec{x}=(0,b,0),\omega) &= \frac{-i}{(2\pi)^{3/2}}\int d^3ke^{ik_2b}\left(\vec{k}-\frac{\omega\epsilon(\omega)}{c^2}\vec{v}\right)\frac{2ze}{\epsilon(\omega)}\frac{\delta(\omega-k_1\vec{v})}{k^2-\omega^2\epsilon(\omega)/c^2} \\ &= \frac{-i2ze}{(2\pi)^{3/2}v\epsilon(\omega)}\int_{-\infty}^{\infty} dk_2\, e^{ik_2b}\int_{-\infty}^{\infty} dk_3 \\ &\qquad \left(\vec{k}-\frac{\omega\epsilon(\omega)}{c^2}\vec{v}\right)\frac{1}{\omega^2/v^2+k_2^2+k_3^2-\omega^2\epsilon(\omega)/c^2},\end{aligned}$$

where $k_1 = \omega/v$. For E_1 this gives

$$\begin{aligned} E_1(\omega) &= \frac{-i2ze\omega}{(2\pi)^{3/2}v^2\epsilon(\omega)}\left(1-\epsilon(\omega)\beta^2\right)\int_{-\infty}^{\infty} dk_2\, e^{ik_2 b} \\ &\qquad \int_{-\infty}^{\infty} dk_3 \frac{1}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2\epsilon(\omega)/c^2} \\ &= \frac{-ize\omega}{\sqrt{2\pi}v^2\epsilon(\omega)}\left(1-\epsilon(\omega)\beta^2\right)\underbrace{\int_{-\infty}^{\infty} dk_2\, e^{ik_2 b}\frac{1}{\sqrt{k_2^2+\lambda^2}}}_{2K_0(\lambda b)} \end{aligned}$$

where

$$\lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2\epsilon(\omega)}{c^2} = \frac{\omega^2}{v^2}\left(1-\beta^2\epsilon(\omega)\right).$$

Note whenever necessary ϵ should be considered to have a positive imaginary part. This can be evaluated[3]

$$E_1(\omega) = -i\sqrt{\frac{2}{\pi}}\frac{ze\omega}{v^2}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)K_0(\lambda b).$$

Next, we turn to E_2 and E_3

$$\begin{aligned} E_2(\omega) &= \frac{-ize}{\sqrt{2\pi}v\epsilon(\omega)}\underbrace{\int_{-\infty}^{\infty} dk_2\, e^{ik_2 b}\, k_2 \frac{1}{\sqrt{\lambda^2+k_2^2}}}_{2i\lambda K_1(\lambda b)} = \frac{ze}{v}\sqrt{\frac{2}{\pi}}\frac{\lambda}{\epsilon(\omega)}K_1(\lambda b) \\ E_3(\omega) &= \frac{-ize}{\sqrt{2\pi}v\epsilon(\omega)}\int_{-\infty}^{\infty} dk_2\, e^{ik_2 b}\int_{-\infty}^{\infty} dk_3 \frac{k_3}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2\epsilon(\omega)/c^2} = 0 \end{aligned}$$

where $E_3 = 0$ by symmetry.

The energy loss due to impact parameters larger than b_0 is

$$\left(\frac{dE}{dx}\right)_{b>b_0} = \int_{b_0}^{\infty} bdb\ \mathrm{Re}\int_0^{\infty} -i\omega\epsilon(\omega)|\vec{E}(\omega)|^2 d\omega$$

[3] Abramowitz and Stegun tell us $K_\nu(xz) = \frac{\Gamma(\nu+\frac12)(2z)^\nu}{\sqrt{\pi}x^\nu}\int_0^\infty \frac{\cos(xt)dt}{(t^2+z^2)^{\nu+\frac12}}$. Expand the cosine in exponentials and rewrite the second term as the extension of the first for $\infty < t < 0$, to get $\int_{-\infty}^{\infty} dx e^{ibx}(x^2+\lambda^2)^{-1/2} = 2K_0(\lambda b)$. The same integral with an extra x (or k_2) in the integrand can be found as the derivative with respect to b, which is $2i\lambda K_1(\lambda b)$, as $K_0'(z) = -K_1(z)$ (9.6.27).

$$\begin{aligned} &= \frac{2}{\pi}\frac{z^2e^2}{v^2}\ \mathrm{Re}\int_0^\infty d\omega\,(-i\omega)\epsilon(\omega)\int_{b_0}^\infty b\,db \\ &\qquad \left[\frac{\omega^2}{v^2}\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)^2 K_0^2(\lambda b) + \underbrace{\frac{\lambda^2}{\epsilon^2(\omega)}}_{\frac{1}{\epsilon(\omega)}\frac{\omega^2}{v^2}\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)} K_1^2(\lambda b)\right] \end{aligned}$$

The term in [ ] is

$$\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)\frac{\omega^2}{v^2\epsilon(\omega)}\left[\left(1-\beta^2\epsilon(\omega)\right)K_0^2 - K_1^2\right]$$

The integral over impact parameter b can be done, as

$$\begin{aligned} \int_a^\infty x\,dx K_0^2(x) &= \frac12 a^2\left(K_1^2(a) - K_0^2(a)\right) \\ \int_a^\infty x\,dx K_1^2(x) &= \frac12 a^2\left(K_0^2(a) - K_1^2(a)\right) + aK_0(a)K_1(a). \end{aligned}$$

I don't quite get this, but Jackson claims

$$\left(\frac{dE}{dx}\right)_{b>b_0} = \frac{2}{\pi}\frac{z^2e^2}{v^2}\ \mathrm{Re}\int_0^\infty d\omega\,(i\omega\lambda^* a)K_1(\lambda^* a)K_0(\lambda a)\left(\frac{1}{\epsilon(\omega)}-\beta^2\right).$$

[Note: the following is my own, Jackson doesn't discuss this.] This evaluation is more accurate than our free electron calculation for large impact parameter, where the atomic electrons feel each others effects by the polarizability, but not for the atomic-scale, so it is best to use this expression only for $b > b_0$, some cutoff, below which we use the free-electron calculation. with ϵ the energy loss corresponding to b_0. From (1),

$$\begin{aligned} d\sigma &= 2\pi bdb = 2\pi\sin\theta d\theta\left(\frac{ze^2}{2vp}\right)^2\frac{1}{\sin^4(\theta/2)} \\ &= 2\pi\frac{dQ^2}{2p^2}\left(\frac{ze^2}{2vp}\right)^2\left(\frac{4p^2}{Q^2}\right)^2 = 2\pi\frac{4p^2}{m}\frac{dT}{T^2}\left(\frac{ze^2}{2vp}\right)^2 \end{aligned}$$

so $b^2 = \left(\frac{2ze^2}{v}\right)^2\frac{1}{2mT}$. So the ϵ we should use for the previous discussion is $\epsilon = \frac{1}{2m}\left(\frac{2ze^2}{vb_0}\right)^2$, with some $b_0 \gg a$, the interatomic distance.