Physics 504, Lecture 14
March 22, 2010

1 Spin and Thomas Precession

Nonrelativistically we know that a current in a loop produces a magnetic field which at large distances corresponds to a magnetic dipole with a magnetic moment given by the current times the area of the loop. In general the magnetic moment (in SI units) is $\vec{m} = \frac{1}{2}\int \vec{r} \times \vec{J}(\vec{r})d^3r$. If the current is due to moving charge $\vec{m} = \frac{1}{2}\int \rho_Q(\vec{r})\vec{r} \times \vec{v}(\vec{r})d^3r$. This looks like the angular momentum except for having the charge density instead of the mass density — if the ratio is a constant,

$$\frac{\rho_Q}{\rho_m} = \frac{Q}{m} \Longrightarrow \vec{m} = \frac{1}{2}\frac{Q}{m}\vec{L} \text{ (SI units) } = \frac{Q}{2mc}\vec{L} \text{ (Gaussian units).}$$

If the ratio is not constant, there will be a correction factor g, and $\vec{m} = g\frac{Q}{2mc}\vec{L}$.

For an elementary particle, say an electron in an atom, the total angular momentum is not just the orbital angular momentum $\vec{L}$ but rather $\vec{L} + \vec{s}$, where $\vec{s}$ is the spin, the intrinsic angular momentum the particle has, which can be thought of as due to the rotation about its center, though this is naïve for a point particle like an electron. The spin can only take on a few discrete values, for an electron $s_z = \pm\frac{1}{2}\hbar$. The magnetic moment of atomic-sized particles is generally called $\vec{\mu}$ rather than $\vec{m}$.

Now the contribution of the orbital motion of an electron in an atom surely has a fixed charge to mass ratio of e/m_e, so we would expect $g = 1$, but the spin is really not understood in classical terms, so it could be otherwise, and according to the Dirac equation $g = 2$, though quantum field theory provides some small corrections (very famously) to that value. These moments can be measured by the splitting of atomic energy levels in a uniform magnetic field, known as the Zeeman effect. If there were no spin, atoms of a given angular momentum $\hbar\ell$ could have components in the direction of $\vec{B}$ given by $\hbar m$, $m = -\ell, -\ell + 1, ..., \ell$ and therefore we should see an odd number of energy levels split by $e\hbar B/2mc$. This is known as the *normal* Zeeman effect. But observations differed, we now know because of elementary particle (electron) spin, and that is called the *anomolous* Zeeman effect.

Let us ask what the equation of motion is for the spin of a particle, whether an electron or an atom. The torque on a dipole in its rest frame is $\vec{\tau} = \vec{m} \times \vec{B}$ and the energy $U = -\vec{m} \cdot \vec{B}$. Thus if $\mathcal{O}'$ is the momentary rest frame of the electron at time t,

$$\frac{d\vec{s}\,'}{dt'} = \frac{ge}{2mc}\vec{s}\,' \times \vec{B}\,', \qquad U' = -\frac{ge}{2mc}\vec{s}\,' \cdot \vec{B}\,'. \tag{1}$$

What is $\vec{B}\,'$? To first order in $\vec{v}/c$, the lorentz transformation $A^\mu{}_\nu = \delta^\mu_\nu - v^\mu \delta^0_\nu/c + v_\nu \delta^\mu_0/c$ so $F'^{\,ij} = F^{ij} - v^i F^{0j}/c + v_j F^{i0}/c \Longrightarrow -\epsilon_{ij\ell} B'_\ell = -\epsilon_{ij\ell} B_\ell + 2v^i E_j/c$, or

$$\vec{B}\,' = \vec{B} - \frac{\vec{v}}{c} \times \vec{E} \qquad\qquad \text{to order } \mathcal{O}(v^2/c^2).$$

If $\mathcal{O}$'s electric field is due to a spherically symmetric potential energy $V(r)$, as for an electron in a hydrogen atom, we have

$$e\vec{E} = -\vec{\nabla} V(\vec{r}) = -\frac{\vec{r}}{r}\frac{dV}{dr},$$

so the spin contribution to the energy *appears to be*

$$\begin{aligned} U' = -\vec{\mu} \cdot \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right) &= -\frac{ge}{2mc}\vec{s}\,' \cdot \vec{B} - \frac{g}{2mc^2}\vec{s}\,' \cdot (\vec{v} \times \vec{r})\,\frac{1}{r}\frac{dV}{dr} \\ &= -\frac{ge}{2mc}\vec{s}\,' \cdot \vec{B} + \frac{g}{2m^2c^2}\vec{s}\,' \cdot \vec{L}\frac{1}{r}\frac{dV}{dr}. \end{aligned} \tag{2}$$

Notice that we have a coupling between the spin and the orbital angular momentum. The rate of change of spin from (1) *appears to be*

$$\frac{d\vec{s}\,'}{dt'} = \frac{ge}{2mc}\vec{s}\,' \times \vec{B} - \frac{g}{2m^2c^2}\vec{s}\,' \times \vec{L}\,\frac{1}{r}\frac{dV}{dr}$$

Unfortunately this is not correct. To get the anomalous Zeeman effect right we need $g = 2$, as the Dirac equation tells us it should be, in the $\vec{B}$ term, but the correct spin orbit term in the fine-structure energy level splittings (2) seems to be half of what we calculated.

The problem here is that what we are doing is trying to find the change in spin by boosting from the lab frame to the rest frame at time t, calculating the change in spin in the rest frame after some infinitesimal times Δt, and then boosting back to the lab frame at time $t+\Delta t$. Both of these boosts were supposed to not involve rotating the coordinate systems, so we don't add a spurious rotation to $\vec{s}$. But the second boost is with a different velocity, so we are actually using a different frame given by $A^\mu{}_\nu(-\vec{v}-\Delta\vec{v})A^\nu{}_\rho(\vec{v})$. Why does that mess up the rotation of the spin? Because

$$\begin{aligned} e^{-(\vec{v}+\Delta\vec{v})\cdot\vec{K}}e^{\vec{v}\cdot\vec{K}} &= \mathbb{1} - \Delta\vec{v}\cdot\vec{K} + \frac{1}{2}v_i(\Delta v)_j[K_i,K_j] + \mathcal{O}(v^2\Delta v) \\ &= \mathbb{1} - \Delta\vec{v}\cdot\vec{K} - \frac{1}{2}\left(\vec{v}\times\Delta\vec{v}\right)\cdot\vec{S}. \end{aligned}$$

The last term represents a rotation of the coordinate system, at a rate $\vec{\omega}_T = -\frac{1}{2}\vec{v}\times\vec{a} = -\frac{1}{2m}\vec{v}\times\vec{F} = -\frac{1}{2m}\vec{r}\times\vec{v}\frac{1}{r}\frac{dV}{dr}$, so even if the spin doesn't rotate in the momentary rest frame, it does rotate in the lab. This needs to be added to the "body-fixed" observed rotation, and as this is of the same form as the $\vec{s}\,'\times\vec{L}$ term, it has the effect of changing g to $g-1$. The same change occurs in the $\vec{S}\cdot\vec{L}$ for the energy[1] term, changing the g of the spin-orbit term to $g-1$. As g is very nearly 2, and was originally expected to be 1, this caused considerable confusion historically.

This argument is sort of clear physically but getting the signs right, as well as going beyond the non-relativistic case, is easier if we formulate things in relativistic (4-D) language.

Now what is the four-dimensional version of the spin? Spin is really an angular momentum, which is a lorentz generator with two spatial indices, but spin is represented by a vector using the 3-dimensional Levi-Civita ϵ. As spin is the angular momentum of a particle in its rest frame, we can use the 4-velocity $U^\alpha = (1,0,0,0)$ in the rest frame to write a 4-vector $S^\alpha = \epsilon^{\alpha\beta\gamma\zeta}U_\beta S_{\gamma\zeta}$, and note that we always have $U_\alpha S^\alpha = 0$, so the spin only has three independent components (classically). So the left hand side of (1) is the spatial part of $dS^\alpha/d\tau$, while the right hand side is $(ge/2mc)\epsilon_{\alpha ij}S^i(-\epsilon_{jk\ell}F^{k\ell}/2) = (ge/2mc)S^\ell F^\alpha{}_\ell$. But this does not determine

[1]Why? The connection of torque and energy is analogous to that of force and energy. $\vec{F} = \frac{d}{dt}\vec{P} = \left[\vec{P},H\right] = -\vec{\nabla}H$ classically, but $-i\left[\vec{P},H\right]$ if we are treating $\vec{P}$ as the quantum-mechanical operator $\vec{P} = -i\hbar\vec{\nabla}$ (with $\hbar=1$). In the same way we have

$$\vec{\tau} = \frac{d}{dt}\vec{L} = -i\left[\vec{L},H\right]_{\rm QM} = -i\vec{L}H,$$

where the second and third $\vec{L}$'s are the differential operator $-i\vec{r}\times\vec{\nabla}$ from lecture 5 (J9.101) the whole of $dS^\alpha/d\tau$ in general, because the equation (1) is not known for the S^0 component in the rest frame, or more generally the component involved in $U_\alpha S^\alpha$, which still needs determining. We can write a correct covariant equation both sides of which vanish when contracted with U_α:

$$\frac{dS^\alpha}{d\tau} - \frac{1}{c^2}U^\alpha U_\beta \frac{dS^\beta}{d\tau} = \frac{ge}{2mc}\left(F^\alpha{}_\beta S^\beta - \frac{1}{c^2}U^\alpha U_\zeta F^\zeta{}_\beta S^\beta\right), \tag{3}$$

because $U^\alpha U_\alpha = c^2$. Fortunately we know that $U_\alpha S^\alpha = 0$ at all times, so $U_\alpha dS^\alpha/d\tau = -(dU_\alpha/d\tau)S^\alpha$, and if the only force on the particle is the Lorentz force[2]

$$\frac{dU_\alpha}{d\tau} = \frac{1}{m}\frac{dp_\alpha}{d\tau} = \frac{1}{m}\frac{e}{c}F_{\alpha\beta}U^\beta.$$

So we can move the second term on the left hand side of (3) to the other side,

$$\begin{aligned}
\frac{dS^\alpha}{d\tau} &= \frac{ge}{2mc}\left(F^\alpha{}_\beta S^\beta - \frac{1}{c^2}U^\alpha U_\zeta F^\zeta{}_\beta S^\beta\right) - \frac{e}{mc^3}U^\alpha S^\beta F_{\beta\gamma}U^\gamma \\
&= \frac{e}{mc}\left[\frac{g}{2}F^\alpha{}_\beta S^\beta + \frac{1}{c^2}\left(\frac{g}{2}-1\right)U^\alpha S_\beta F^{\beta\zeta}U_\zeta\right] \qquad (4)
\end{aligned}$$

In the rest frame $S'^\mu = (0, \vec{s})$, so applying a Lorentz transformation

$$S^0 = \gamma\vec{\beta}\cdot\vec{s}, \qquad \vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}(\vec{\beta}\cdot\vec{s})\vec{\beta}.$$

(This is most easily seen by assuming $\vec{v}$ is in the x direction, and then making the result rotationally invariant.) To compare to the previous discussion of $d\vec{s}/dt$, which we expect to be good to first order in v or β, we need to be careful, because while $\vec{s} \approx \vec{S}$ to first order, the time derivative $d\vec{\beta}/d\tau = e\vec{E}/mc$ is zeroth order. Thus to first order in v,

$$\frac{d\vec{s}}{d\tau} = \frac{d\vec{S}}{d\tau} - \frac{e}{2mc^2}\left[(\vec{s}\cdot\vec{E})\vec{v} + (\vec{s}\cdot\vec{v})\vec{E}\right].$$

The first term can be evaluated to first order from (4)

$$\begin{aligned}
\frac{dS_i}{d\tau} &= \frac{e}{mc}\left[\frac{g}{2}F^i{}_j s_j + \frac{g}{2}F^i{}_0\vec{\beta}\cdot\vec{s} - \frac{1}{c^2}\left(\frac{g}{2}-1\right)v_i s_j F^{j0}c\right] \\
\frac{d\vec{S}}{d\tau} &= \frac{e}{mc}\left[\frac{g}{2}\vec{s}\times\vec{B} + \frac{g}{2}(\vec{v}\cdot\vec{s})\vec{E}/c - \frac{1}{c}\left(\frac{g}{2}-1\right)\vec{v}(\vec{s}\cdot\vec{E})\right].
\end{aligned}$$

[2] If there is a gradient in the magnetic field, there is an additional force from the coupling of a magnetic moment with an inhomogeneous field. We assume that is not present in this discussion.

Putting the terms together, to first order in v,

$$\begin{aligned}
\frac{d\vec{s}}{d\tau} &= \frac{e}{mc}\left[\frac{g}{2}\vec{s}\times\vec{B}+\frac{g}{2}(\vec{v}\cdot\vec{s})\vec{E}/c-\frac{1}{c}\left(\frac{g}{2}-1\right)\vec{v}(\vec{s}\cdot\vec{E})\right] \\
&\qquad -\frac{e}{2mc^2}\left[(\vec{s}\cdot\vec{E})\vec{v}+(\vec{s}\cdot\vec{v})\vec{E}\right] \\
&= \frac{e}{mc}\vec{s}\times\left[\frac{g}{2}\vec{B}-\frac{g-1}{2c}\vec{v}\times\vec{E}\right]. \qquad (5)
\end{aligned}$$

Notice that the $\vec{E}$ term has g replaced by $g-1$ compared to the naïve derivation. The full expression, correct for all $\vec{v}$, follows from (4), but is quite complicated.

A very interesting thing happens if $g=2$, as predicted by the Dirac equation. Then we have the second term in (4) vanishing, and if we have a pure magnetic field, so that $F^{0\mu}=0$, we see that $S^0=\gamma\vec{\beta}\cdot\vec{s}$ is a constant. But so are γ and $|\beta|$ in a pure magnetic field, so we see that the helicity, $\hat{\beta}\cdot\vec{s}$ is conserved.

Now in quantum field theory there are small corrections to $g=2$. Because only the corrections contribute to the evolution of the helicity, they can be measured very precisely. Also extreme effort has been invested in the theoretical calculations. Experiment says

$$\frac{g-2}{2} = \begin{array}{r} 0.001\,159\,652\,180\,73 \\ \pm 0.000\,000\,000\,000\,28 \end{array}$$

and theory says[3]

$$\frac{g-2}{2} = \begin{array}{r} 0.001\,159\,652\,182\,79 \\ \pm 0.000\,000\,000\,007\,71 \end{array}$$

certainly one of the most accurately measured quantities in physics.

[3] Aoyama, Hayakawa, Kinoshita, Nio, Phys. Rev. D 77, 053012 (2008)