Physics 504, Lecture 4

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1 Curvilinear Coordinates

Many of our notions in "vector calculus" are developed in the context of cartesian coordinates describing a Euclidean space. We will use $r^i, i = 1, 2, \dots D$ as cartesian coordinates describing a D dimensional Euclidean space. (Note the index on the coordinates has been written as a superscript rather than as a subscript — this is in preparation for discussing Minkowski space soon and curved spaces, possibly, later in your career.) Usually we will take $D = 3$. Being a Euclidean space, the distance δs between the points labelled with $\{r^i\}$ and $\{r^i + \delta r^i\}$ is given by Pythagoras:

$$(\delta s)^2 = \sum_i (\delta r^i)^2.$$

Vectors are discribed in terms of unit vectors $\hat{e}_i$, and a displacement is a vector $\Delta\vec{r} = \sum_i \Delta r^i \hat{e}_i$. Scalar and vector fields can be considered functions of $\vec{r}$ or functions of the D coordinates $\{r^i\}$, and the gradiant and laplacian of a scalar function $\Phi(\vec{r})$, and the divergence and curl of a vector function $\vec{V}(\vec{r})$ are given by

$$\begin{aligned}
\vec{\nabla}\Phi &= \sum_i \frac{\partial \Phi}{\partial r^i}\hat{e}_i, \\
\vec{\nabla}\cdot\vec{V} &= \sum_i \frac{\partial V_i}{\partial r^i}, \\
\nabla^2\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi &= \sum_i \frac{\partial^2 \Phi}{\partial r^{i\,2}}, \\
\vec{\nabla}\times\vec{V} &= \sum_{ijk} \epsilon_{ijk}\frac{\partial V_k}{\partial r^j}\hat{e}_i,
\end{aligned}$$

where the last expression applies only in $D = 3$ and uses the totally antisymmetric tensor ϵ_{ijk}, which in Euclidean space with Cartesian components takes the value[1] $\epsilon_{123} = 1$.

[1]In differential geometry, or in the discussion of forms without the restriction to orthonormal basis vectors, one introduces the Levi-Civita symbol ε_{ijk} (with D subscripts in D dimensional space), which is proportional to, but not equal to, the flat-space ϵ_{ijk} for which $\epsilon_{123} = 1$. The relation is $\varepsilon_{ijk} = \sqrt{|\det g_{\cdot\cdot}|}\epsilon_{ijk}$. The volume element is then given by the D-form $\sum \varepsilon_{\mu_1,\dots\mu_D} dq^{\mu_1} \wedge \cdots \wedge dq^{\mu_D}$. But in this lecture we will only use ϵ_{ijk}, not ε_{ijk}. Note the notational distinction I have made is not standard.

But there are many situations in which it is useful to use coordinates other than cartesian, not to mention spaces which are other than Euclidean. So we need to develop the expressions appropriate to these generalized coordinates for vectors and these differential operators.

A generalized coordinate system $q^i, i = 1, \dots D$ is a parameterization of the space, that is, we do assume that the $\{q^i\}$ describe the space, so that any point described by $\vec{r}$ in cartesian coordinates may also be described in the new system by $\{q^i = q^i(\vec{r})\}$, and each triplet (q^1, q^2, q^3) (or D-plet) in some allowed range specifies a unique point $\vec{r}(q^1, q^2, q^3)$. This map must be one-to-one, at least locally[2] at a generic point. This requires the Jacobian $\det(\partial q^i/\partial r^j) \neq 0$.

The first step is to describe the distance between two points close to each other. If we consider the distance ds between two infinitesimally separated points P and P' described by q^i and $q^i + \delta q^i$ in generalized coordinates and by $\vec{r}$ and $\vec{r} + \delta\vec{r}$ in cartesian coordinates, we may write

$$(\delta s)^2 = \sum_k (\delta r^k)^2 = \sum_k \left(\sum_i \frac{\partial r^k}{\partial q^i}\delta q^i\right)\left(\sum_j \frac{\partial r^k}{\partial q^j}\delta q^j\right) = \sum_{ij} g_{ij}\delta q^i \delta q^j,$$

where

$$g_{ij} = \sum_k \frac{\partial r^k}{\partial q^i}\frac{\partial r^k}{\partial q^j}.$$

This real symmetric matrix is known as the **metric tensor**. It is in general a nontrivial function of the position, $g_{ij}(q)$.

To repeat: the metric tensor determines the distance between neighboring points,

$$(\delta s)^2 = \sum_{ij} g_{ij}\delta q^i \delta q^j.$$

A scalar function on three dimensional space may be given as a function of the cartesian coordinates $f(\vec{r})$ or of the generalized coordinates $\tilde{f}(q^i)$. As

[2]Many generalized coordinate systems have singular points and domains so that the q^i's are not defined over the entire real line. For example, in spherical coordinates $r \geq 0$, and at $r = 0$, all values of θ and ϕ correspond to the same point.

this is supposed to be actually a function of the point rather than of the coordinates used to describe the point, we must have $\tilde{f}(q) = f(\vec{r}(q))$.

The gradient of a scalar function is a vector function, a vector at each point of the space. It is the same vector at a given physical point regardless of which coordinate system we use, but the expression of this vector will depend on the basis vectors used. We do not expect to use the cartesian basis vectors $\hat{e}_i$ with the generalized coordinates, so we ask how to define, in general, the appropriate basis vectors "in the q^i direction". For instance, we might define the unit vector

$$\tilde{e}_1 = \lim_{\delta q^1 \to 0} \frac{\vec{r}(q^1 + \delta q^1, q^2, q^3) - \vec{r}(q^1, q^2, q^3)}{\delta s} = \sum_k \frac{\partial r^k}{\partial q^1} \hat{e}_k / \sqrt{g_{11}},$$

and similarly for the other $\tilde{e}_i$ In general, however, these will not be orthogonal, as

$$\tilde{e}_1 \cdot \tilde{e}_2 = \sum_k \frac{\partial r^k}{\partial q^1} \frac{\partial r^k}{\partial q^2} / \sqrt{g_{11} g_{22}} = g_{12} / \sqrt{g_{11} g_{22}},$$

which need not be zero.

This is awkward, but things are even worse. The direction specified above is not, in general, the same as the normal to the contour surface of constant q^1 values, for $\tilde{e}_2$ is tangent to that surface. This awkwardness is handled in two different ways. In the discussion of curved spaces, as dealt with in differential geometry and general relativity, one simply drops the idea of expressing the gradient of a scalar as a vector field expanded in terms of unit basis vectors. In fact, the gradient of a scalar is considered a 1-form, dual to rather than the same as a vector field. Forms are discussed later. In discussing curvilinear coordinates for flat three dimensional space, it is more appropriate to put a limitation on the kinds of coordinates we will discuss: we will limit ourselves to systems of coordinates for which the contour surfaces of different coordinates are orthogonal whereever they cross. Then their normals, the directions $\vec{\nabla} q^i$ and $\vec{\nabla} q^j$, are orthogonal for $i \neq j$. Such coordinates are called orthogonal curvilinear coordinates. Let us define in general

$$g^{ij} := \vec{\nabla} q^i \cdot \vec{\nabla} q^j = \sum_k \frac{\partial q^i}{\partial r^k} \frac{\partial q^j}{\partial r^k}.$$

Note that g^{ij} is not the same as g_{ij}. In fact

$$\sum_\ell g^{i\ell} g_{\ell j} = \sum_\ell \sum_k \frac{\partial q^i}{\partial r^k} \frac{\partial q^\ell}{\partial r^k} \sum_m \frac{\partial r^m}{\partial q^\ell} \frac{\partial r^m}{\partial q^j} = \sum_{km} \frac{\partial q^i}{\partial r^k} \delta_{km} \frac{\partial r^m}{\partial q^j} = \delta_{ij},$$

so $g^{\cdot\cdot}$ is the inverse matrix to $g_{\cdot\cdot}$. For orthogonal curvilinear coordinates, $g^{\cdot\cdot}$ is diagonal, which is true if and only if $g_{\cdot\cdot}$ is diagonal.

As $g_{\cdot\cdot}$ is, in general, a positive definite matrix, if it is diagonal, the diagonal elements must be positive, so for an orthogonal coordinate system, we can rewrite $g_{ij} = h_i^2 \delta_{ij}$, and $g^{ij} = h_i^{-2} \delta_{ij}$. Then the unit vectors are

$$\tilde{e}_i = h_i^{-1} \sum_k \hat{e}_k \frac{\partial r^k}{\partial q^i} =: \sum_k B_{ki} \hat{e}_k \tag{1}$$

with the inverse relation

$$\hat{e}_k = \sum_i h_i \tilde{e}_i \frac{\partial q^i}{\partial r^k} =: \sum_i A_{ki} \tilde{e}_i = \sum_i A_{ki} \sum_\ell B_{\ell i} \hat{e}_\ell. \tag{2}$$

$\perp$ 2/2 As the $\hat{e}_k$ are independent, this implies $\sum_i A_{ki} B_{\ell i} = \delta_{k\ell}$ or $AB^T = 1\!\!1$.

As the set of basis vectors $\hat{e}_k$ and the set $\tilde{e}_i$ are each orthonormal, the matrix $A_{ki} = h_i \partial q^i / \partial r^k$ which connects them is an orthgonal matrix, and $B = A$. This orthogonality can be easily verified as[3]

$$\sum_k A_{ki} A_{kj} = h_i h_j \sum_k (\partial q^i / \partial r^k)(\partial q^j / \partial r^k) = h_i h_j g^{ij} = \delta_{ij}.$$

Thus A can be written two ways,

$$A_{ki} = h_i \frac{\partial q^i}{\partial r^k} = h_i^{-1} \frac{\partial r^k}{\partial q^i}.$$

1.1 Vector Fields

A vector function on space can be expressed in terms of any set of basis vectors we choose, even if we choose different basis vectors at different points, which is what we are doing in using the $\tilde{e}_i$. For example, in spherical coordinates $\tilde{e}_r$ always points away from the origin, so that the $\tilde{e}_r$'s at different angles are not parallel, and therefore not equal[4]. The vector itself, at a given

[3]Note that in the expression $h_i h_j g^{ij}$ there is no sum (which would usually be implied) over i and j. The extra h_i factors in these expositions in terms of unit vectors do not neatly fit with the usual relativity summation conventions.

[4]Notice that we are assuming a notion of "parallel transport" of the vectors which is built into the definition of Euclidean space, and which says that two vectors are equal if their components in *cartesian* coordinates are equal, not if their components in generalized curvilinear coordinates are equal. In differential geometry one studies intrinsically curved spaces, in which the simple Euclidean concept of transporting a vector parallel to itself becomes a much more complicated idea.

point, is the same regardless of which basis is used to describe it, though its components are different. Thus

$$\vec{V} = \sum_k V_k \hat{e}_k = \sum \tilde{V}_i \tilde{e}_i,$$

with the components in curvilinear coordinates $\tilde{V}_i$ determined in terms of the cartesian coordinates from the connection of the basis vectors,

$$\hat{e}_k = \sum_i A_{ki} \tilde{e}_i, \quad \tilde{V}_i = \sum_k V_k A_{ki}, \quad V_k = \sum_i A_{ki} \tilde{V}_i.$$

1.2 Derivatives

The gradient can be rewritten

$$\begin{aligned}
\vec{\nabla} f &= \sum_k \frac{\partial f}{\partial r^k} \hat{e}_k = \sum_{k\ell m} \frac{\partial \tilde{f}}{\partial q^\ell} \frac{\partial q^\ell}{\partial r^k} h_m \tilde{e}_m \frac{\partial q^m}{\partial r^k} = \sum_{\ell m} \frac{\partial \tilde{f}}{\partial q^\ell} h_m \tilde{e}_m g^{m\ell} \\
&= \sum_{\ell m} \frac{\partial \tilde{f}}{\partial q^\ell} h_m \tilde{e}_m h_m^{-2} \delta_{m\ell} = \sum_m h_m^{-1} \frac{\partial \tilde{f}}{\partial q^m} \tilde{e}_m,
\end{aligned}$$

or

$$\vec{\nabla} f = \sum_m h_m^{-1} \frac{\partial \tilde{f}}{\partial q^m} \tilde{e}_m. \tag{3}$$

Because we like to think of the function f as the same thing as $\tilde{f}$, in the sense that they take the same values at the same physical point, we often neglect to put the tilde on and write $f(q)$

1.2.1 Velocity

While not a vector field, the velocity of a particle is still important, and we need to know how to express it in curvilinear coordinates. We assume the coordinate system itself is time-independent[5]. Then by the chain rule, we have

$$\begin{aligned}
\vec{v} &= \sum_k \frac{dr^k}{dt} \hat{e}_k = \sum_k \left(\sum_i \frac{\partial r^k}{\partial q^i} \frac{dq^i}{dt} \right) \left(\sum_j h_j \frac{\partial q^j}{\partial r^k} \tilde{e}_j \right) = \sum_{ij} \frac{dq^i}{dt} h_j \delta_{ij} \tilde{e}_j \\
&= \sum_j h_j \frac{dq^j}{dt} \tilde{e}_j.
\end{aligned}$$

[5]Thus we are excluding here a discussion of rotating coordinate systems.

Note it is $h_j dq^j$ which has the right dimensions for an infinitesimal *length*, while dq^j by itself might not.

To consider a particular example, let's take spherical coordinates. As the contours of r, θ and ϕ are spherical shells centered on the origin, cones with their vertices at the origin, and planes including the z-axis, which are "obviously" orthogonal[6], this is an orthogonal curvilinear coordinate system. Looking graphically at the lengths spanned by changing one of the coordinates while keeping the others fixed, and comparing to $(ds)^2 = h_r^2(dr)^2 + h_\theta^2(d\theta)^2 + h_\phi^2(d\phi)^2$, we see that

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta.$$

Thus $\vec{v} = \dot{r}\tilde{e}_r + r\dot{\theta}\tilde{e}_\theta + r\sin\theta\dot{\phi}\tilde{e}_\phi$ and $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$.

1.2.2 Derivatives of vectors

Even in cartesian coordinates, the spatial derivative of a vector field is complicated, as it involves two directions, the direction of partial differentiation and that of the vector. In curvilinear coordinates it is further complicated by the fact that the basis vectors are themselves varying from one point to another. Thus in differentiating the vector we must also take into account the changes in $\tilde{e}_i$, for while in cartesian coordinates

$$\frac{\partial \hat{e}_i}{\partial r^j} = 0,$$

in curvilinear orthogonal coordinates we have

$$\begin{aligned}
h_j^{-1} \frac{\partial}{\partial q^j} \tilde{e}_i &= h_j^{-1} \sum_{k\ell} \frac{\partial r^k}{\partial q^j} \frac{\partial}{\partial r^k} \left(h_i^{-1} \hat{e}_\ell \frac{\partial r^\ell}{\partial q^i} \right) = h_j^{-1} \sum_{k\ell} \frac{\partial r^k}{\partial q^j} \frac{\partial A_{\ell i}}{\partial r^k} \hat{e}_\ell \\
&= \sum_{k\ell} A_{kj} \frac{\partial A_{\ell i}}{\partial r^k} \hat{e}_\ell.
\end{aligned}$$

We see that the expression is not transparent, and in fact we shall find a simpler expression by considering differential forms.

[6]If this is not obvious, you can laboriously calculate $\vec{\nabla} r \cdot \vec{\nabla} \theta$, *etc.*

1.3 Vector fields and forms

One way of thinking about vector fields in Euclidean space is in terms of differential forms, in particular 1-forms. An arbitrary 1-form on $\mathbb{R}^3$ can be written in terms of the basis 1-forms dq^i as $\omega = \sum_i A_i dq^i$, but if we are using orthogonal curvilinear coordinates to describe Euclidean space, it is more common to write the coordinates as multipliers of the normalized[7] 1-forms $\omega_i = h_i dq^i$, so $\omega = \sum_i A_i dq^i = \sum_i \tilde{V}_i \omega_i$. We may associate the vector $\vec{V}$ with the one form ω if they have the same $\tilde{V}$'s. Note that if $\omega = df = \sum_i (\partial f/\partial q^i) dq^i = \sum_i h_i^{-1}(\partial f/\partial q^i)\omega_i$, we see (Eq. 3) that the associated vector is $V = \vec{\nabla} f$, the gradient of f.

In three dimensional space, a 2-form can also be associated with a vector, because an arbitrary two form $\omega^{(2)} = \frac{1}{2}\sum_{ij} B_{ij}\omega_i \wedge \omega_j$ can be associated with the vector $\vec{B}$ whose coefficients $\tilde{B}_i = \frac{1}{2}\sum_{jk}\epsilon_{ijk}B_{jk}$, and $B_{jk} = \sum_i \epsilon_{ijk}\tilde{B}_i$. Notice that if $\vec{V}$ is associated with $\omega^{(1)}$ and $\omega^{(2)} = d\omega^{(1)}$,

$$\begin{aligned}\omega^{(2)} &= \frac{1}{2}\sum_{ij} B_{ij}\omega_i \wedge \omega_j = d(\sum_i \tilde{V}_i h_i dq^i) = \sum_{ij}\frac{\partial(\tilde{V}_i h_i)}{\partial q^j} dq^j \wedge dq^i \\ &= \sum_{ij} h_i^{-1} h_j^{-1} \frac{\partial(\tilde{V}_i h_i)}{\partial q^j}\omega_j \wedge \omega_i,\end{aligned}$$

so, using the antisymmetry of $\omega_j \wedge \omega_i$, we see that

$$\frac{1}{2}B_{ij} = \frac{1}{2}h_i^{-1}h_j^{-1}\left(\frac{\partial(\tilde{V}_j h_j)}{\partial q^i} - \frac{\partial(\tilde{V}_i h_i)}{\partial q^j}\right) = \frac{1}{2}\sum_k \epsilon_{ijk}\tilde{B}_k$$

or

$$\tilde{B}_k = \frac{1}{2}\sum_{ij}\epsilon_{ijk}\frac{1}{h_i h_j}\left(\frac{\partial}{\partial q^i}\tilde{V}_j h_j - \frac{\partial}{\partial q^j}\tilde{V}_i h_i\right) = \sum_{ij}\epsilon_{ijk}\frac{1}{h_i h_j}\frac{\partial}{\partial q^i}\left(\tilde{V}_j h_j\right).$$

If the q^i are cartesian coordinates all the $h_i = 1$ and we recognize $\vec{B} = \vec{\nabla}\times\vec{V}$, which is a coordinate independent statement, so we have derived the formula for a curl in general orthogonal curvilinear coordinates.

[7] When we write vectors and forms in terms of dq^i or $\partial_i = \partial/\partial q^i$, the up-or-down placement of indices is significant, but when we use normalized basis vectors this becomes unclear, and we could have written ω^i just as well.

Finally we ask what happens if we apply the exterior derivative operator d to the 2-form associated with a vector $\vec{B}$,

$$\begin{aligned} d\left(\frac{1}{2}\sum_{ijk}\epsilon_{ijk}\tilde{B}_i\omega_j\wedge\omega_k\right) &= d\left(\frac{1}{2}\sum_{ijk}\epsilon_{ijk}\tilde{B}_i h_j h_k dq^j \wedge dq^k\right) \\ &= \frac{1}{2}\sum_{ijk}\epsilon_{ijk}\frac{\partial(\tilde{B}_i h_j h_k)}{\partial q^i}dq^i\wedge dq^j\wedge dq^k \\ &= \frac{1}{2}\sum_{ijk}\epsilon_{ijk}\frac{1}{h_i h_j h_k}\frac{\partial(\tilde{B}_i h_j h_k)}{\partial q^i}\omega_i\wedge\omega_j\wedge\omega_k \\ &= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q^i}\left(\tilde{B}_i\frac{h_1h_2h_3}{h_i}\right)\omega_1\wedge\omega_2\wedge\omega_3.\end{aligned}$$

A 3-form $\omega^{(3)}$ can be associated with a scalar function f by

$$\begin{aligned}\omega^{(3)} &= f dr^1\wedge dr^2\wedge dr^3 = \frac{1}{6}f\sum_{abc}\epsilon_{abc}dr^a\wedge dr^b\wedge dr^c \\ &= \frac{1}{6}f\sum_{abcijk}\epsilon_{abc}\frac{\partial r^a}{\partial q^i}\frac{\partial r^b}{\partial q^j}\frac{\partial r^c}{\partial q^k}dq^i\wedge dq^j\wedge dq^k \\ &= \frac{1}{6}f\sum_{abcijk}\epsilon_{abc}\left(h_i^{-1}\frac{\partial r^a}{\partial q^i}\right)\left(h_j^{-1}\frac{\partial r^b}{\partial q^j}\right)\left(h_k^{-1}\frac{\partial r^c}{\partial q^k}\right)\omega_i\wedge\omega_j\wedge\omega_k \\ &= \frac{1}{6}f\det A\sum_{ijk}\epsilon_{ijk}\omega_i\wedge\omega_j\wedge\omega_k \\ &= f\det A\;\omega_1\wedge\omega_2\wedge\omega_3 \\ &= f\;\omega_1\wedge\omega_2\wedge\omega_3,\end{aligned}$$

where in the last step we assumed the curvilinear coordinates were ordered so that $\tilde{e}_1\times\tilde{e}_2 = +\tilde{e}_3$, or equivalently that A is not only orthogonal but also has determinant $+1$.

Examining the function associated with $\omega^{(3)} = d\omega^{(2)}$ using cartesian coordinates q^i, we see that $f = \sum_i \partial\tilde{B}_i/\partial q^i = \vec{\nabla}\cdot\vec{B}$, and as this is a coordinate independent statement, we see that for general orthogonal curvilinear coordinates

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i}\tilde{B}_i\right).$$

Finally, we can evaluate the Laplacian on a scalar,

$$\begin{aligned} f &= \nabla^2\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i}h_i^{-1}\frac{\partial\Phi}{\partial q^i}\right) \\ &= \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\Phi}{\partial q^i}\right) \end{aligned}$$

This last operator maps a scalar into a scalar, and is therefore independent of worries about establishing unit vectors for the curvilinear coordinates. Thus there is a more general form,

$$\nabla^2 = \frac{1}{\sqrt{g}}\sum_{ij}\frac{\partial}{\partial q^i}g^{ij}\sqrt{g}\frac{\partial}{\partial q^j},$$

where $g := \det g_{\cdot\cdot}$. This form holds for any coordinate system in a Riemannian space, not just orthogonal curvilinear coordinates in Euclidean space.

Applying the above to spherical coordinates, we have

$$\begin{aligned} \vec{\nabla}f &= \frac{\partial f}{\partial r}\tilde{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\tilde{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\tilde{e}_\phi, \\ \vec{\nabla}\times\vec{V} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}r\sin\theta V_\phi - \frac{\partial}{\partial\phi}rV_\theta\right)\tilde{e}_r \\ &\quad + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}V_r - \frac{\partial}{\partial r}r\sin\theta V_\phi\right)\tilde{e}_\theta \\ &\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}rV_\theta - \frac{\partial}{\partial\theta}V_r\right)\tilde{e}_\phi \\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta V_\phi\right) - \frac{\partial}{\partial\phi}V_\theta\right]\tilde{e}_r \\ &\quad + \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V_r - \frac{1}{r}\frac{\partial}{\partial r}\left(rV_\phi\right)\right]\tilde{e}_\theta \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rV_\theta\right) - \frac{\partial}{\partial\theta}V_r\right]\tilde{e}_\phi, \\ \vec{\nabla}\cdot\vec{B} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\tilde{B}_r\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\tilde{B}_\theta\right) + \frac{\partial}{\partial\phi}\left(r\tilde{B}_\phi\right)\right) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tilde{B}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\tilde{B}_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tilde{B}_\phi \\ \nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}r^2\sin\theta\frac{\partial\Phi}{\partial r} + \frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{\partial}{\partial\phi}\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}. \end{aligned}$$