Dispersion

As the differential equations we have been dealing with are linear and without time-dependent parameters, Fourier transform from time to frequency results in equations for which frequency is simply a parameter, and the question of dispersion, that is, the dependence of ϵ and μ on frequency, does not complicate the discussion. But upon transforming back into the time domain, the fact that in general there is dispersion has important effects.

A model for $\epsilon(\omega)$

One model for ϵ which works surprisingly well is to model the electrons has damped harmonic oscillators, so

$$m\left(\ddot{\vec{x}}+\gamma\dot{\vec{x}}+\omega_0^2\vec{x}\right)=-e\vec{E}(\vec{x},t)$$

and ignore magnetic forces. Here ω_0 is the resonant frequeny of the oscillator, independent of the frequency of oscillation of the fields.

The cause of the damping force $m\gamma\dot{\vec{x}}$ will be considered much later, in Chapter 16. The amplitude of oscillation we would expect to be on atomic scales, so for frequencies for which $k \sim \omega/c$ is small compared to $10^9/\text{m}$, or $\omega << 10^{17}/\text{s}$, we may ignore the $\vec{x}$ dependence of $\vec{E}$. Taking a harmonic $\vec{E} \propto e^{-i\omega t}$, the dipole moment is

$$\vec{p} = -e\vec{x} = \frac{e^2}{m}\,\frac{\vec{E}}{\omega_0^2 - \omega^2 - i\omega\gamma}.$$

If there are N molecules per unit volume and Z electrons per molecule, we have a polarization of $\vec{P} = NZ\vec{p}$ and $\epsilon/\epsilon_0 = 1 + \frac{NZe^2}{\epsilon_0 m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$. Actually we generally assume the Z electrons behave like diffenent oscillators, with f_i electrons/molecule oscillating with frequency ω_i and damping factor γ_i. The f_i, which sum to Z, are called *the oscillator strengths*. So our model for the dielectric constant is

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_i \frac{f_i}{\omega_0^2 - \omega^2 - i\omega\gamma}.$$

Though our model is purely classical its applicability involves the quantum mechanical states of the electrons in the molecule. The frequencies are related to those of photons given off in transitions from one level to another, and the damping factors γ, which have units of frequency, are related to the decay widths of those levels, generally far smaller than ω_i. Thus it has an effect only for ω very close to transition frequencies, which tend to be in the optical range $\omega_i \sim 10^{15}/\text{ s}$. Away from that region ϵ is very close to being real, and for $\omega < \omega_i$, $\epsilon(\omega)$ slowly increases, though it threatens to diverge as $\omega \to \omega_i$, while for ω greater than any of the oscillator frequencies, $\epsilon < 1$, though it approaches 1 as $\omega \to \infty$.

This behavior for ω, with Re ϵ increasing, is called *normal dispersion*, the reverse *anomalous dispersion*.

Consider a plane wave $\vec{E}e^{ikz-i\omega t}$ with $k=\sqrt{\epsilon\mu}\omega$. We are assuming no magnetizability, so $\mu=\mu_0$ is constant and real, but with Re $\epsilon>0$, Im $\epsilon>0$, we have $k=\beta+i\alpha/2$, with β and α real and positive. The imaginary part contributes a damping factor $e^{-\alpha z/2}$ to the fields, or $e^{-\alpha z}$ to the power in the fields. α is called the *attenuation constant* or *absorption coefficient*, due to dissipation of energy in the damping terms of the oscillators. Note $k^2=\beta^2+i\alpha\beta-\alpha^2/4=\epsilon\mu_0\omega^2=\frac{\omega^2}{c^2}\frac{\epsilon}{\epsilon_0}$, so

$$\beta^2-\frac{\alpha^2}{4}=\frac{\omega^2}{c^2}\frac{\text{Re }\epsilon}{\epsilon_0},\qquad \beta\alpha=\frac{\omega^2}{c^2}\frac{\text{Im }\epsilon}{\epsilon_0}.$$

Usually $\alpha\ll\beta$ and $\alpha\approx\dfrac{\text{Im }\epsilon}{\text{Re }\epsilon}\beta$, and $\beta=\sqrt{\dfrac{\text{Re }\epsilon}{\epsilon_0}}\dfrac{\omega}{c}$.

Conduction

With the charges coupled to springs, a constant electric field can only pull them so far, and there will be no DC current produced. But if one of the spring constants goes to zero, say with oscillator strength f_0, that term contributes $iNe^2 f_0/m\omega(\gamma_0 - i\omega)$ to the sum, so we have

$$\epsilon(\omega) = \epsilon_b + i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)},$$

where

$$\epsilon_b = \epsilon_0 + \frac{Ne^2}{m}\sum_{i\neq 0}\frac{f_i}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

includes the polarization of all the other oscillators.

Suppose there are no “free” charges or currents, though we are including these zero-mode charges as part of the polarization.

Then Maxwell's equation for the harmonic fields

$$\vec{\nabla} \times \vec{H} = -i\omega\vec{D} = -i\omega\left(\epsilon_b + i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}\right)\vec{E}.$$

Now we may reinterpret the zero-mode charges as free, so they are no longer considered part of the polarization, and redefine $\vec{D} = \epsilon_b\vec{E}$, and the other term as $\vec{J} = \sigma\vec{E}$, with a conductivity

$$\sigma = \frac{f_0 Ne^2}{m(\gamma_0 - i\omega)}.$$

Consider copper, with $N = 8.5 \times 10^{28}$ atoms/m^3 and assume one zero-mode (or free) electron per atom (its electronic structure is $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^1$, the last its valence electron), and its conductivity at low frequencies is $\sigma = 5.9 \times 10^7/\Omega \cdot \mathrm{m}$, so we may calculate $\gamma_0 = 4.0 \times 10^{13}/\mathrm{s}$. Thus for frequencies well into the hundreds of gigahertz region (microwaves) we may treat the conductivity as constant and real.

High Frequency

At the other end of the frequency spectrum, for $\omega \gg \omega_j$ and $\omega \gg \gamma_j$, all oscillators behave the same and we have $\epsilon = \epsilon_0 - NZe^2/m\omega^2$, or

$$\frac{\epsilon}{\epsilon_0} = 1 - \frac{\omega_P^2}{\omega^2}, \quad \text{with the plasma frequency} \quad \omega_P^2 = \frac{NZe^2}{\epsilon_0 m}.$$

From $k = \omega\sqrt{\epsilon\mu} = \frac{\omega}{c}\sqrt{\epsilon/\epsilon_0} = \frac{1}{c}\sqrt{\omega^2 - \omega_P^2}$, (assuming $\mu = \mu_0$).

A relation between k and ω is called a dispersion relation, with $\omega = \omega(k)$.

In general the dispersion relation $\omega(k) = \sqrt{\omega_P^2 + c^2k^2}$ holds only for frequencies higher than the oscillator frequencies.

But for free electrons those frequencies are zero, so we might look at $\omega < \omega_P$, in which case the wave number $k = \frac{1}{c}\sqrt{\omega^2 - \omega_P^2}$ becomes imaginary, and an incident electromagnetic wave $e^{ikz} = e^{-|k|x}$ cannot penetrate, but is attenuated exponentially with distance with $\alpha_P \approx 2\omega_P/c$ for low frequencies. This is also what happens for a metal with visible light, incident light cannot penetrate and is reflected, which is why they are shiny.

We will skip Jackson 7.5E, 7.6, 7.7

Group Velocity

Earlier we saw that the constant-phase surfaces of a monochromatic plane wave move at the phase velocity $v_p = \omega/k = c/n$, and so if $n < 1$, which does happen above the plasma frequency, this velocity can be greater than the velocity of light in vacuum. Does this violate special relativity?

Not so simple

- monochromatic wave is infinite in spatial extent, can't tell if it moves.
- A sharp wave pulse contains wide spread of ω's. With dispersion, it will spread out, not just move with some velocity.
- Best you can do is look at a wave packet limited in space and with limited range of frequencies.

Consider a pulse consisting of a narrow spread of wave numbers $k = k_0 + \kappa$, with amplitude $A(k)$ contributing only for small κ,

$$\begin{aligned} u(x,t) &= \int dk A(k) e^{ikx - i\omega(k)t} \\ &\approx e^{ik_0x - i\omega(k_0)t} \int d\kappa A(k_0+\kappa) e^{i\kappa\left[x - (d\omega/dk)|_{k_0} t\right]}, \end{aligned}$$

The phase outside the integral does not affect the shape of the packet, which we see travels with the *group velocity*

$$v_g(k_0) = \left.\frac{d\omega}{dk}\right|_{k_0}.$$

In general the index of refraction of a dielectric material is frequency dependent, and as $\omega(k) = ck/n = ck/n(\omega)$,

$$v_g(k_0) = \frac{d\omega}{dk} = \left(\frac{d}{d\omega}\frac{\omega n(\omega)}{c}\right)^{-1} = \frac{c}{n(\omega) + \omega(dn/d\omega)}.$$

As $dn/d\omega$ is generally positive except right near the absorption frequencies, the $\omega dn/d\omega$ term in the denominator generally decreases the group velocity compared to the phase velocity.

Even above the plasma frequency, with $ck = \sqrt{\omega^2 - \omega_P^2}$, we have $cdk = \omega d\omega/\sqrt{\omega^2 - \omega_P^2} > d\omega$, so $v_g = d\omega/dk < c$.

Let's look at a movie of a wave packet.

We will skip Jackson 7.9