To review, in our original presentation of Maxwell's equations, ρ_{all} and $\vec{J}_{\text{all}}$ represented all charges, both "free" and "bound". Upon separating them, "free" from "bound", we have (dropping quadripole terms):

- For the electric field
  - $\vec{E}$ called *electric field*
  - $\vec{P}$ called *electric polarization* is induced field
  - $\vec{D}$ called *electric displacement* is field of "free charges"
  - $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$
- For the magnetic field
  - $\vec{B}$ called *magnetic induction* (unfortunately)
  - $\vec{M}$ called *magnetization* is the induced field
  - $\vec{H}$ called *magnetic field*
  - $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

Then the two Maxwell equations with sources, Gauss for $\vec{E}$ and Ampère, get replaced by

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{J} \end{aligned}$$

Energy in the Fields

The rate of work done by E&M fields on charged particles is:

$$\sum q_j \vec{v}_j \cdot \vec{E}(\vec{x}_j, t),$$

or if we describe it by current density,

$$\int_V \vec{J} \cdot \vec{E}.$$

This must be the rate of loss of energy U in the fields themselves, so

$$-\frac{dU}{dt} = \int_V \vec{J} \cdot \vec{E}.$$

Now by Ampère's Law,

$$\vec{J} \cdot \vec{E} = \left(\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{E}.$$

From the product rule and the cyclic nature of the triple product, we have for any vector fields

$$\vec{\nabla}\cdot(\vec{V}\times\vec{W}) = \vec{W}\cdot(\vec{\nabla}\times\vec{V}) - \vec{V}\cdot(\vec{\nabla}\times\vec{W}),$$

so we may rewrite $\left(\vec{\nabla}\times\vec{H}\right)\cdot\vec{E}$ as

$$-\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{H}\cdot\vec{\nabla}\times\vec{E} = -\vec{\nabla}\left(\vec{E}\times\vec{H}\right) - \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$$

where we used Faraday's law. Thus

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{J}\cdot\vec{E} = 0.$$

Let us assume *the medium is linear without dispersion in electric and magnetic properties*, that is $\vec{B}\propto\vec{H}$ and $\vec{D}\propto\vec{E}$. Then let us propose that the *energy density of the fields* is

$$u(\vec{x},t) = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right),$$

we have

$$\frac{\partial u}{\partial t} + \vec{J} \cdot \vec{E} + \vec{\nabla} \cdot \left(\vec{E} \times \vec{H} \right) = 0.$$

As this is true for any volume, we may interpret this equation, integrated over some volume V with surface ∂V as saying that the rate of increase in the energy in the fields plus the energy of the charged particles plus the flux of energy out of the volume is zero, that is, no energy is created or destroyed. The flux of energy is then given by the Poynting vector

$$\vec{S} = \vec{E} \times \vec{H}.$$

We have made assumptions which only fully hold for the vacuum, as we assumed linearity and no dispersion (the ratio of $\vec{D}$ to $\vec{E}$ independent of time).

Linear Momentum

So as not to worry about such complications, let's restrict our discussion to the fundamental description, or alternatively take our medium to be the vacuum, with the fields interacting with distinct charged particles (m_j, q_j at $\vec{x}_j(t)$).

The mechanical linear momentum in some region of space is $\vec{P}_{\rm mech} = \sum_j m_j \dot{\vec{x}}_j$, so

$$\begin{aligned} \frac{d\vec{P}_{\rm mech}}{dt} &= \sum_j \vec{F}_j = \sum_j q_j \left(\vec{E}(\vec{x}_j) + \vec{v}_j \times \vec{B}(\vec{x}_j) \right) \\ &= \int_V \rho \vec{E} + \vec{J} \times \vec{B}, \end{aligned}$$

provided no particles enter or leave the region V.

Let us postulate that the electromagnetic field has a linear momentum density

$$\vec{g} = \frac{1}{c^2} \vec{E} \times \vec{H} = \epsilon_0 \vec{E} \times \vec{B}.$$

Then the total momentum inside the volume V changes at the rate

$$\frac{d\vec{P}_{\text{Tot}}}{dt} = \int_V \rho\vec{E} + \vec{J}\times\vec{B} + \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}).$$

Using Maxwell's laws to substitute $\epsilon_0\vec{\nabla}\cdot\vec{E}$ for ρ and $\epsilon_0\left(c^2\vec{\nabla}\times\vec{B} - \partial\vec{E}/\partial t\right)$ for $\vec{J}$,

$$\begin{aligned}
\frac{d\vec{P}_{\text{Tot}}}{dt} &= \epsilon_0\int_V \vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2(\vec{\nabla}\times\vec{B})\times\vec{B} + \vec{B}\times\frac{\partial\vec{E}}{\partial t} \\
&\qquad -\frac{\partial}{\partial t}\left(\vec{B}\times\vec{E}\right) \\
&= \epsilon_0\int_V \vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2(\vec{\nabla}\times\vec{B})\times\vec{B} - \frac{\partial\vec{B}}{\partial t}\times\vec{E} \\
&= \epsilon_0\int_V \vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2(\vec{\nabla}\times\vec{B})\times\vec{B} - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right).
\end{aligned}$$

For any vector field $\vec{V}$,

$$\begin{aligned}\left(\vec{V} \times (\vec{\nabla} \times \vec{V}\right)_\alpha &= \sum_{\beta\gamma\mu\nu} \epsilon_{\alpha\beta\gamma} V_\beta \, \epsilon_{\gamma\mu\nu} \frac{\partial}{\partial x_\mu} V_\nu \\ &= \frac{1}{2} \frac{\partial V^2}{\partial x_\alpha} - \sum_\beta V_\beta \frac{\partial V_\alpha}{\partial x_\beta} \\ &= -\sum_\beta \frac{\partial}{\partial x_\beta} \left(V_\alpha V_\beta - \frac{1}{2} V^2 \delta_{\alpha\beta} \right).\end{aligned}$$

So we see that the E terms in dP/dt may be written as

$$\frac{\partial}{\partial x_\beta} \epsilon_0 \left(E_\alpha E_\beta - \frac{1}{2} \vec{E}^2 \delta_{\alpha\beta} \right).$$

The same may be done for the magnetic field, as the missing $\vec{\nabla} \cdot \vec{B}$ is zero. Thus we define

The Maxwell Stress Tensor

$$T_{\mu\nu} = \epsilon_0 \left[E_\mu E_\nu - \frac{1}{2}\vec{E}^{\,2}\delta_{\mu\nu} + c^2 \left(B_\mu B_\nu - \frac{1}{2}\vec{B}^{\,2}\delta_{\mu\nu} \right) \right],$$

which is called the *Maxwell stress tensor*. Then

$$\left(\frac{d\vec{P}_{\text{Tot}}}{dt} \right)_\mu = \sum_\nu \int_V \frac{\partial}{\partial x_\nu} T_{\mu\nu}.$$

By Gauss's law, the integral of this divergence over V is the integral of $\sum_\beta T_{\alpha\beta}\hat{n}_\beta$ over the surface ∂V of the volune considered, so $T_{\alpha\beta}\hat{n}_\beta$ is the flux of the α component of momentum out of the surface.

Complex Fields, Dispersion

In linear media we can assume $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$, but actually this statement is only good for the Fourier transformed (in time) fields, because all media (other than the vacuum) exhibit dispersion, that is, the permittivity ϵ and magnetic permeability μ depend on frequency. So we need to deal with the Fourier transformed fields[1]

$$\begin{aligned}\vec{E}(\vec{x},t) &= \int_{-\infty}^{\infty} d\omega \vec{E}(\vec{x},\omega)e^{-i\omega t} \\ \vec{D}(\vec{x},t) &= \int_{-\infty}^{\infty} d\omega \vec{D}(\vec{x},\omega)e^{-i\omega t}\end{aligned}$$

and we define linear permittivity as $D(\vec{x},\omega) = \epsilon(\omega)\vec{E}(\vec{x},\omega)$, and similarly $B(\vec{x},\omega) = \mu(\omega)\vec{H}(\vec{x},\omega)$. The inverse fourier transform does not like multiplication — if $\epsilon(\omega)$ is not constant, we do not have $\vec{D}(\vec{x},t)$ proportional to $\vec{E}(\vec{x},t)$.

[1] Note the 2π inconsistency with 6.33, and slide 8 of lecture 1.

Note that the electric and magnetic fields in spacetime are supposed to be real, not complex, valued. From

$$\begin{aligned}\vec{E}(\vec{x},t) &= \int_{-\infty}^{\infty} d\omega \vec{E}(\vec{x},\omega)e^{-i\omega t} \\ = \vec{E}^*(\vec{x},t) &= \int_{-\infty}^{\infty} d\omega \vec{E}^*(\vec{x},\omega)e^{i\omega t} = \int_{-\infty}^{\infty} d\omega \vec{E}^*(\vec{x},-\omega)e^{-i\omega t}\end{aligned}$$

which tells us $\vec{E}(\vec{x},\omega) = \vec{E}^*(\vec{x},-\omega)$, and similarly for the other fields. Thus the permittivity and permeability also obey

$$\epsilon(-\omega) = \epsilon^*(\omega), \qquad \mu(-\omega) = \mu^*(\omega).$$

The power transferred to charged particles includes an integral of

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \iint d\omega d\omega' \vec{E}^*(\omega')(-i\omega\epsilon(\omega)) \cdot \vec{E}(\omega)e^{-i(\omega-\omega')t},$$

where we took the complex conjugate expression for $\vec{E}(t)$, but alternatively we could have taken the complex conjugate expression for $\vec{D}$ and interchanged ω and ω',

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \iint d\omega d\omega' \vec{E}^*(\omega')(i\omega'\epsilon^*(\omega')) \cdot \vec{E}(\omega) e^{-i(\omega-\omega')t},$$

or averaging the two expressions

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{i}{2} \iint d\omega d\omega' \vec{E}^*(\omega') \cdot \vec{E}(\omega) [\omega'\epsilon^*(\omega') - \omega\epsilon(\omega)] e^{-i(\omega-\omega')t},$$

We are often interested in the situation where the fields are dominately near a given frequency, and if we ignore the rapid oscillations in this expression which come from one ω positive and one negative, we may assume the ω's differ by an amount for which a first order variation of ϵ is enough to consider, so
$\omega'\epsilon^*(\omega') \approx \omega\epsilon^*(\omega) + (\omega' - \omega) d(\omega\epsilon^*(\omega))/d\omega$, and

$$[i\omega'\epsilon^*(\omega') - i\omega\epsilon(\omega)] = 2\omega \text{Im } \epsilon(\omega) - i(\omega - \omega') \frac{d}{d\omega} (\omega\epsilon^*(\omega))$$

Inserting this back into $\vec{E}\cdot\partial\vec{D}/\partial t$, the $-i(\omega-\omega')$ can be interpreted as a time derivative, so

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \iint d\omega d\omega' \vec{E}^*(\omega')\cdot\vec{E}(\omega)\omega \mathrm{Im}\ \epsilon(\omega)e^{-i(\omega-\omega')t}$$

$$+\frac{\partial}{\partial t}\,\frac{1}{2}\iint d\omega d\omega' \vec{E}^*(\omega')\cdot\vec{E}(\omega)\frac{d}{d\omega}\Big(\omega\epsilon^*(\omega)\Big)e^{-i(\omega-\omega')t}.$$

If ϵ were pure real, the first term would not be present, and if ϵ were constant, the $d[\omega\epsilon^*(\omega)]/d\omega$ would be ϵ, consistent with the $u=\frac{1}{2}\epsilon E^2$ we had in our previous consideration. There is, of course, a similar result from the $\vec{H}\cdot d\vec{B}/dt$ term. More generally, we can think of the first term as energy lost to the motion of bound charges within the molecules, not included in $\vec{J}\cdot\vec{E}$, which goes into heating the medium, while the second term describes the energy density in the macroscopic electromagnetic fields.

Harmonic Fields

We are often interested in considering fields oscillating with a given frequency, so

$$\vec{E}(\vec{x},t) = \text{Re}\ \left(\vec{E}(\vec{x})e^{-i\omega t}\right) = \frac{1}{2}\left[\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}^{*}(\vec{x})e^{i\omega t}\right].$$

If we have another such field, say $\vec{J}(\vec{x},t)$, the dot product is

$$\begin{aligned}
&\vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t) \\
&= \frac{1}{4}\left[\left(\vec{J}(\vec{x})e^{-i\omega t} + \vec{J}^{*}(\vec{x})e^{i\omega t}\right)\cdot\left(\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}^{*}(\vec{x})e^{i\omega t}\right)\right] \\
&= \frac{1}{2}\text{Re}\ \left[\vec{J}^{*}(\vec{x})\cdot\vec{E}(\vec{x}) + \vec{J}(\vec{x})\cdot\vec{E}(\vec{x})e^{-2i\omega t}\right].
\end{aligned}$$

The second term is rapidly oscillating, so can generally be ignored, and the average product is just half the product of the two harmonic (fourier transformed) fields, one complex conjugated.

Energy in Harmonic Fields

Fourier transforming Maxwell's equations, for the Harmonic fields, for which $\partial/\partial t$ becomes $-i\omega$, we see

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$$
$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\times\vec{H} + i\omega\vec{D} = \vec{J}$$

The current induced will also be harmonic, so the power lost to current is

$$\frac{1}{2}\int d^3x\vec{J}^*\cdot\vec{E}$$
$$= \frac{1}{2}\int d^3x\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}^* - i\omega\vec{D}^*\right)$$
$$= \frac{1}{2}\int d^3x\left[-\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^*\right) + \left(\vec{\nabla}\times\vec{E}\right)\cdot H^* - i\omega\vec{E}\cdot\vec{D}^*\right]$$
$$= \frac{1}{2}\int d^3x\left[-\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^*\right) + i\omega\vec{B}\cdot H^* - i\omega\vec{E}\cdot\vec{D}^*\right]$$

We define the complex Poynting vector $\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$ and the harmonic densities of electric and magnetic energy, $w_e = \frac{1}{4}\vec{E} \cdot \vec{D}^*$ and $w_m = \frac{1}{4}\vec{B} \cdot \vec{H}^*$, and thus find (using Gauss's divergence law)

$$\frac{1}{2}\int_V \vec{J}^* \cdot \vec{E} + 2i\omega \int_V (w_e - w_m) + \oint_{\partial V} \vec{S} \cdot \hat{n} = 0,$$

which, as a complex equation, contains energy flow. If the medium may be taken as pure lossless dielectrics and magnetics, with real ϵ and μ, the real part of this equation says

$$\frac{1}{2}\int_V \mathrm{Re}\,\left(\vec{J}^* \cdot \vec{E}\right) + \oint_{\partial V} \mathrm{Re}\,\left(\vec{S} \cdot \hat{n}\right) = 0,$$

which says that the power transferred to the charges and that flowing out of the region, on a time averaged basis, is zero. But there is also an oscillation of energy between the electric and magnetic fields. For example, with a pure plane wave in vacuum, with $\vec{E}$ and $\vec{H}$ in phase, S is real, $\vec{J}$ vanishes, and the imaginary part tells us the electric and magnetic energy densities are equal.

Rotational and PCT properties

Let us briefly describe how our fields behave under rotational symmetry, reflection, charge conjugation and time reversal.

From mechanics, forces are vectors, so as a charge experiences a force $q\vec{E}$, $\vec{E}$ is a vector under rotations, unchanged under time reversal, reverses in sign under charge conjugation, and under parity $\vec{E} \to -\vec{E}$, as a proper vector should.[2]

Charge and charge density are proper scalars, reversing under charge conjugation (by definition).

Velocity is a proper vector, so for $q\vec{v} \times \vec{B}$ to be a proper vector as well, $\vec{B}$ must be a pseudovector, whose components are unchanged under parity, because the cross product (and $\epsilon_{\alpha\beta\gamma}$) are multiplied by -1 under parity. Of course we also have $\vec{B}$ odd under charge conjugation.

[2] under reflection in a mirror, the component perpendicular to the mirror reverses, but parity includes a rotation of 180° about that axis, reversing the other two components as well.

Under time-reversal, forces and acceleration are unchanged ($\propto d^2/dt^2$) but velocity changes sign, so $\vec{E}$ (and $\vec{P}$ and $\vec{D}$) are even under time-reversal, but $\vec{B}$ (and $\vec{M}$ and $\vec{H}$) are odd under time-reversal, that is, the get multiplied by -1.
Maxwell's equations, the Lorentz force law, and all the other formulae we have written are consistent with symmetry under rotations and P,C, and T reflections. That means the laws themselves are invariant under these symmetries.

Magnetic Monopoles

In vacuum, Maxwell's equations treat $\vec{E}$ and $\vec{B}$ almost identically. Then if we consider a doublet $\vec{\mathcal{D}} = \begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}$ the two Gauss' laws say

$$\vec{\nabla} \cdot \vec{\mathcal{D}} = 0, \qquad \vec{\nabla} \times \mathcal{D} + i\sigma_2 \frac{1}{c}\frac{\partial \vec{\mathcal{D}}}{\partial t} = 0,$$

where $i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This not only lets us write 4 laws as 2, but shows that the equations would be unchanged by a rotation in this two dimensional space. But this symmetry is broken by our having observed electric charges and currents but no magnetic charges (magnetic monopoles). But maybe we just haven't found them yet, and we should add them in.

Maxwell with Monopoles

Then Maxwell's equations become

$$\vec{\nabla} \cdot \vec{D} = \rho_e, \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_e$$

$$\vec{\nabla} \cdot \vec{B} = \rho_m, \qquad -\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + \vec{J}_m$$

From these equations and our previous symmetry properties, we see that ρ_e is a scalar, ρ_m is a pseudoscalar, $\vec{J}_e$ a proper vector and $\vec{J}_m$ a pseudovector. Define the doublets[3]

$$\vec{\mathcal{H}} = \begin{pmatrix} \epsilon_0^{\frac{1}{2}} \vec{E} \\ \mu_0^{\frac{1}{2}} \vec{H} \end{pmatrix} \quad \vec{\mathcal{B}} = \begin{pmatrix} \mu_0^{\frac{1}{2}} \vec{D} \\ \epsilon_0^{\frac{1}{2}} \vec{B} \end{pmatrix} \quad \vec{\mathcal{J}} = \begin{pmatrix} \mu_0^{\frac{1}{2}} \vec{J}_e \\ \epsilon_0^{\frac{1}{2}} \vec{J}_m \end{pmatrix} \quad \mathcal{R} = \begin{pmatrix} \mu_0^{\frac{1}{2}} \rho_e \\ \epsilon_0^{\frac{1}{2}} \rho_m \end{pmatrix}$$

so Maxwell's equations become

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = \mathcal{R}, \qquad i\sigma_2 \vec{\nabla} \times \mathcal{H} = \frac{\partial \vec{\mathcal{B}}}{\partial t} + \vec{\mathcal{J}}.$$

[3]The $\sqrt{\epsilon_0}$ and $\sqrt{\mu_0}$ are due to the unfortunate SI units we are using. Let $Z_0 = \sqrt{\mu_0/\epsilon_0}$.

Thus we see that the equations are invariant under a simultaneous rotation of these doublets in the two dimensional space, in particular

$$\vec{E} \to \vec{E}\cos\xi + Z_0\vec{H}\sin\xi.$$

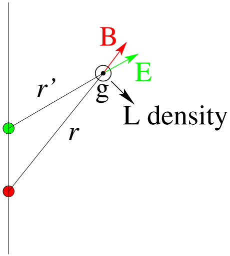
But this is very pecular, as $\vec{E}$ is a proper vector and $\vec{H}$ is a pseudovector. So such an object would not have a well defined parity.

Quantization of Charge

Dirac noticed that a point charge and a point monopole, both at rest, gives a momentum density and an angular momentum density in the electromagnetic fields. Consider a monopole at the origin, $\rho_m = g\delta(\vec{x})$, and a point charge $\rho_e = q\delta(\vec{x} - (0, 0, D))$ on the z axis a distance D away.

We have magnetic and electric fields

$$\vec{H} = \frac{g}{4\pi\mu_0}\frac{\hat{e}_r}{r^2}, \quad \vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{e}_{r'}}{r'^2}.$$



The momentum density in the fields is $\vec{g} = \frac{1}{c^2}\vec{E} \times \vec{H}$, which points in the aximuthal direction, $\propto \hat{e}_\phi$ in spherical coordinates. As the magnitude is independent of ϕ, when we integrate this vector in ϕ, we will get zero, so the total momentum vanishes.

But the angular momentum density, $\vec{r} \times \vec{g}$, has a component everywhere in the $-z$ direction, so its integral will not vanish.

$$\begin{aligned}
\vec{L} &= \frac{1}{c^2}\int d^3r\, \vec{r} \times (\vec{E} \times \vec{H}) = \frac{g}{4\pi\mu_0 c^2}\int d^3r\, \frac{\vec{r} \times (\vec{E} \times \vec{r})}{r^3} \\
&= \frac{g\epsilon_0}{4\pi}\int d^3r\, \frac{r^2\vec{E} - \vec{r}(\vec{r}\cdot\vec{E})}{r^3} = \frac{g\epsilon_0}{4\pi}\int d^3r\, (\vec{E}\cdot\vec{\nabla})\frac{\vec{r}}{r} \\
&= \frac{g\epsilon_0}{4\pi}\left(\oint_{\partial V} \hat{n}\cdot\vec{E}\frac{\vec{r}}{r} - \int_V \frac{\vec{r}}{r}\vec{\nabla}\cdot\vec{E}\right)
\end{aligned}$$

Taking the volume to be a large sphere, the first term vanishes because $\hat{n}\cdot\vec{E}$ is constant$/r^2$, so the integral is $\propto \int d\Omega\hat{n} = 0$. In the second term $\vec{\nabla}\cdot\vec{E} = q\delta^3(\vec{r} - (0,0,D)/\epsilon_0$ so overall

$$\vec{L} = -\frac{qg}{4\pi}\hat{e}_z.$$

Charge Quantization

There are several interesting things about this result. One is that it is independent of how far apart the two objects are. But even more crucial, if we accept from quantum mechanics the requirement that angular momentum is quantized in usits of $\hbar/2$, we see that if just one monopole of magnetic charge g exists anywhere, all purely electric charges must be

$q_n = nh/g$ where n is an integer. Furthermore, as there are electrons, the smallest nonzero monopole has a "charge" at least h/e, and the Coulomb force between two such charges would be

$$\frac{g^2}{4\pi\mu_0} \Big/ \frac{e^2}{4\pi\epsilon_0} = \frac{h^2\epsilon_0}{e^4\mu_0} = \left(\frac{h\epsilon_0 c}{e^2}\right)^2 = \left(\frac{137}{2}\right)^2 \sim 4700$$

times as big as the electric force between two electrons at the same separation.

Vector Potential and Monopoles

The ability to define vector and scalar potentials to represent the electromagnetic fields depended on the two sourceless Maxwell equations. If we have monopoles, these conditions don't apply whereever a monopole exists, so that $\vec{B}$ is not divergenceless everywhere. Even if $\vec{\nabla} \cdot \vec{B}$ fails to vanish at only one point, it means $\vec{B}$ cannot be written as a curl throughout space. Poincaré's Lemma tells us it is possible on a contractible domain, which is not true for a sphere surrounding the monopole. One can define $\vec{A}$ consistently everywhere other than on a "Dirac string" extending from the monopole to infinity, but $\vec{A}$ is not defined on the string.