Lecture 23 April 22, 2010

Bremsstrahlung:

We have seen that whenever charged particles accelerate, radiation will be produced, $\propto (\dot{\vec{v}})^2$, so the quicker the momentum change the more the energy emitted.

Atomic scattering is quick!

As we saw, in passing through material, charged particles to lose energy, like stepping on the brakes. The radiation produced is called **brems** (brakes) **strahlung** (radiation).

Fast electrons produce gamma rays, which scatter and produce eletron-positron pairs, a shower of energy in counters which allows experimentalists to measure the energy of the initial electron.

Radiation from a sudden collision

Last time we found the general formula for the Fourier transform of $\sqrt{c/4\pi}R\vec{E}(t)$:

$$\vec{A}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega R/c} \int_{-\infty}^{\infty} e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)} \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\right] dt.$$

In an atomic collision, the change in $\vec{\beta}$ takes place over a very short time interval in a very localized region. So if ω/c is small, the phase $\exp i\omega(t-\hat{n}\cdot\vec{r}(t)/c)$ is essentially a constant, and can be dropped, and then the integrand is a total derivative, with

$$\begin{aligned}\vec{A}(\omega) &\sim \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\right] dt \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta}_f)}{1-\hat{n}\cdot\vec{\beta}_f} - \frac{\hat{n}\times(\hat{n}\times\vec{\beta}_i)}{1-\hat{n}\cdot\vec{\beta}_i}\right),\end{aligned}$$

where $\vec{\beta}_f$ and $\vec{\beta}_i$ are the final and initial velocities of the particle.

Projecting onto a specific polarization $\vec{\epsilon}$, which is $\perp \hat{n}$, we have the intensity

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c} \left| \vec{\epsilon}^* \cdot \left(\frac{\vec{\beta}_f}{1 - \hat{n} \cdot \vec{\beta}_f} - \frac{\vec{\beta}_i}{1 - \hat{n} \cdot \vec{\beta}_i} \right) \right|^2 .$$

Disturbing features of this formula

Notice that this formula for the intensity per frequency range is independent of frequency! Troubles at both ends:

- Total energy radiated per solid angle diverges as $\omega \to \infty$

  $$\int_0^\infty d\omega \frac{d^2 I}{d\omega d\Omega} = \infty.$$

  We made approximation $\omega/c \ll$ atomic size, so need a cutoff for high ω. Not disturbing.
- Low frequencies: The energy in a range $[\omega, \omega + d\omega]$ is independent of ω. No divergence in energy from lower limit — why is this disturbing?

Quantum Mechanical interpretation

We have done a classical calculation of the energy emitted in radiation, but the world is quantum mechanical. We know we are really emitting photons, each with energy $\hbar\omega$. How many?

$$\frac{d^2N}{d\omega d\Omega} = \frac{1}{\hbar\omega}\frac{d^2I}{d\omega d\Omega} \propto \frac{1}{\omega}$$

so the total number of photons emitted $\sim \int_0 d\omega/\omega = \infty$.

A classical calculation predicting radiating ΔE in some $d\omega d\Omega$ means an *expected value* $\Delta E/\hbar\omega$ for the number of photons in that range. Fractional values are okay, because this is a probabilistic calculation.

In a Quantum Mechanical scattering process, we ask what the amplitude is for going from a given initial state to a given final state. For example, we might ask for the transition of an electron and proton from momenta $\vec{p}_i$ and $\vec{P}_i$ to $\vec{p}_f$ and $\vec{P}_f$, with $\vec{p}_i + \vec{P}_i = \vec{p}_f + \vec{P}_f$.

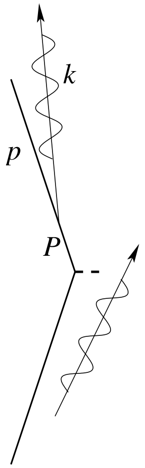
Infrared divergence

But now we see it is zero! Because we expect infinitely many photons to be in the final state! This becomes a problem in quantum field theory, where one does perturbative calculations in powers of the electron charge, and where the lowest approximation for a scattering $\vec{p}_i + \vec{P}_i \to \vec{p}_f + \vec{P}_f$. like the is perfectly reasonable, but the next correction gives nonsense.

One needs to change the question: What is the probability $\vec{p}_i + \vec{P}_i \to \vec{p}_f + \vec{P}_f +$ arbitrary numbers of photons with a total energy less than some small amount δE. This turns out to be finite and calculable.

Features of Bremsstrahlung

If our charged particle is ultrarelativistic, the radiation will be strongly peaked in two directions, $\hat{n} \parallel \vec{\beta}_i$ and $\hat{n} \parallel \vec{\beta}_f$. Recall that a uniformly moving charge has an electric field which moves along with it. The field keeps going even after the charge gets bumped. And the absence of the appropriate field for $\vec{\beta}_f$ is also radiation.



If k^α is the four-momentum of the emitted photon, and the charged particle has mass m and momenta p^α, either before or after the collision, the "virtual particle" with momentum P^α consisting of the final charge plus the photon (or the initial charge minus the photon) has an invariant mass

$$M^2 = \frac{1}{c^2}\left(p^\alpha \pm k^\alpha\right)^2 = m^2 \pm \frac{2}{c^2} p^\alpha k_\alpha = m^2 \pm \frac{m|k|}{c} \gamma(1 - \hat{n} \cdot \vec{\beta}).$$

In quantum field theory Feynman diagrams like this have factors of $1/(P^2 - m^2)$ for virtual particles, so we see that the $1 - \hat{n} \cdot \vec{\beta}$ factors are coming from almost-on-shell propagators.

Beta decay

$$Z \to (Z \pm 1) + e^{\mp} + \nu.$$

Nucleus is heavy, so effectively a charge has instanteously accelerated from zero to $\vec{\beta}$.

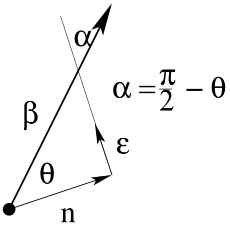
$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \left| \frac{\vec{\epsilon}^* \cdot \vec{\beta}}{1 - \hat{n} \cdot \vec{\beta}} \right|^2 .$$

Again we have no ω dependence, though there is a cutoff. Our instantaneous assumption requires $\omega/c \ll$ nuclear size, but also energy conservation requires $\hbar\omega < (m_Z - m_{Z\pm 1} - m_e)c^2$.

Bremsstrahlung in β decay

The radiation in the $\hat{n}$ direction is polarized, as the component perpendicular to the plane containing $\hat{n}$ and $\vec{\beta}$ is not excited (by $\vec{\epsilon}^* \cdot \vec{\beta}$). Then, as we see in the diagram, for the direction that *is* excited, $\vec{\epsilon}^* \cdot \vec{\beta} = \beta \sin\theta$,

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \beta^2 \frac{\sin^2\theta}{\left(1 - \hat{n}\cdot\vec{\beta}\right)^2}.$$



Integrating over all angles,
$\int d\Omega = \int \sin\theta d\theta d\phi = 2\pi \int_{-1}^{1} du$, we have

$$\begin{aligned} \frac{dI}{d\omega} &= \frac{e^2}{4\pi^2 c} 2\pi \int_{-1}^{1} \frac{\beta^2(1-u^2)}{(1-\beta u)^2} du \\ &= \frac{e^2}{2\pi\beta c} \int_{1-\beta}^{1+\beta} \frac{\beta^2 - 1 + 2x - x^2}{x^2} dx \\ &= \frac{e^2}{\pi c} \left[\frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} - 2 \right]. \end{aligned}$$

For small β,

$$\frac{dI}{d\omega} = \frac{e^2}{\pi c}\left[\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right] = \frac{2e^2\beta^2}{3\pi c} + \mathcal{O}(\beta^4),$$

so multiplying by the cutoff $\Delta E/\hbar$ we see the fraction of rest energy loss going into radiation is $\dfrac{2e^2\beta^2}{3\pi\hbar c} = \dfrac{2\beta}{3\pi}\alpha$, where $\alpha = \dfrac{e^2}{\hbar c} \approx \dfrac{1}{137}$ is the fine structure constant. So the emitted energy is quite small but interesting.
Why? People are very interested in the neutrino possibly having a small mass. One way to look is to find the upper limit on the electron energy,

$$E_e \leq (m_Z - m_{Z\pm1} - m_\nu)c^2.$$

But this assumes no radiation, so the energy distribution of electrons needs to be corrected for energy lost to radiation for accurate detection of the upper limit. No one has found it yet.

Electron Capture

The reverse process is electron capture. A nucleus of charge Ze grabs an inner electron, converts a proton to a neutron, and emits a neutrino

$$Z + e^- \rightarrow (Z-1) + \nu.$$

Classical Picture:
Nonrelativistic electron in circle of radius a, $\perp z$, with angular frequency ω_0, up until time $t = 0$, at which point it disappears. For $t < 0$:

$$\begin{aligned} \vec{r}(t) &= a\cos(\omega_0 t + \alpha)\hat{e}_x + a\sin(\omega_0 t + \alpha)\hat{e}_y, \\ \vec{v}(t) &= -\omega_0 a\sin(\omega_0 t + \alpha)\hat{e}_x + \omega_0 a\cos(\omega_0 t + \alpha)\hat{e}_y. \end{aligned}$$

Integrating today's initial equation by parts

$$\vec{A}(\omega) = -\sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega R/c} \int_{-\infty}^{\infty} \frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}} \frac{d}{dt}\left[e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)}\right] dt.$$

Concentrate on low enough frequencies so $\omega\hat{n}\cdot\vec{r}(t) < a\omega$ is negligible.

As we have assumed nonrelativistic electrons, $\beta \ll 1$, we will also drop the denominator. Thus

$$A(\omega) = -i\omega\sqrt{\frac{q^2}{8\pi^2 c}}e^{i\omega R/c}\int_{-\infty}^{0} \hat{n}\times(\hat{n}\times\vec{\beta})e^{i\omega t}dt.$$

Assume we are observing from an angle θ wrt z, so let $\hat{n} = \sin\theta\hat{e}_x + \cos\theta\hat{e}_z$, and let $\hat{e}_\perp$ $(= \hat{e}^y)$ and $\hat{e}_\parallel$ be the unit polarization vectors perpendicular to and within, respectively, the plane containing $\hat{n}$ and $\hat{e}_z$.
We need $\hat{n}\times(\hat{n}\times\vec{v})$ which is found from $\hat{n}\times\vec{v} = v_x\cos\theta\hat{e}_\perp + v_y\hat{e}_\parallel$, $\hat{n}\times(\hat{n}\times\vec{v}) = -v_x\cos\theta\hat{e}_\parallel + v_y\hat{e}_\perp$,
The integrals are given in terms of[1]

$$I_1 = \int_{-\infty}^{0}\cos(\omega_0 t+\alpha)e^{i\omega t} = \frac{-i\omega\cos\alpha - \omega_0\sin\alpha}{\omega^2-\omega_0^2}$$

$$I_2 = \int_{-\infty}^{0}\sin(\omega_0 t+\alpha)e^{i\omega t} = \frac{\omega_0\cos\alpha - i\omega\sin\alpha}{\omega^2-\omega_0^2}$$

[1] We have damped out the $t \to -\infty$ limit.

Thus

$$\begin{aligned}\frac{d^2 I}{d\omega d\Omega} &= \frac{e^2\omega^2\omega_0^2 a^2}{4\pi^2 c^3}\left(|I_2\cos\theta)|^2 + |I_1|^2\right)\\ &= \frac{e^2\omega^2\omega_0^2 a^2}{4\pi^2 c^3(\omega^2-\omega_0^2)^2}\Big(\omega^2\cos^2\alpha + \omega_0^2\sin^2\alpha\\ &\qquad + \cos^2\theta(\omega_0^2\cos^2\alpha + \omega^2\sin^2\alpha)\Big)\end{aligned}$$

The position in the circle at the moment of absorbtion is random, so we average over the value of α, to get

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2\omega_0^2 a^2}{4\pi^2 c^3(\omega^2-\omega_0^2)^2}\frac{\left(\omega^2+\omega_0^2\right)\left(1+\cos^2\theta\right)}{2}$$

If we don't know in which direction the electron was rotating we should average over angles, or if we want the total power radiated per frequency range, we should integrate over $d\Omega$:

$$\frac{dI}{d\omega} = \int d\Omega \frac{d^2 I}{d\omega d\Omega} = \frac{2e^2}{3\pi c}\left(\frac{\omega_0 a}{c}\right)^2 \frac{\omega^2\left(\omega^2+\omega_0^2\right)}{(\omega^2-\omega_0^2)^2}.$$

Quantum interpretation

Electron is captured by nucleus at $r = 0$, so only s-orbital electrons can be captured, usually K shell $(n = 1, \ell = 0)$. They don't exactly go around in circles. But perhaps a 2p electron will jump down so quickly $(< 1/\omega)$ that the fields won't notice it wasn't the one gobbled up. So take the 2p radius as a and the energy of the $2p \to 1s$ transition, $3Z^2e^2/8a_0$ for $\hbar\omega_0$, where $a_0 = \hbar^2/me^2 = 53$ pm is the Bohr radius. The 2p radius is roughly $a = a_0/Z$. Thus the expected number of photons emitted per energy is

$$N(\hbar\omega) = \frac{dI(\omega)}{\hbar d\omega} \Big/ \hbar\omega \approx \frac{3}{32\pi} Z^2 \left(\frac{e^2}{\hbar c}\right)^3 \frac{1}{\hbar\omega} \left[\frac{\omega^2(\omega^2+\omega_0^2)}{\left(\omega^2-\omega_0^2\right)^2}\right].$$

$\hbar\omega N(\hbar\omega) \underset{\omega\to\infty}{\longrightarrow} 3\alpha^3 Z^2/32\pi$. No infrared problem. What about $\omega \approx \omega_0$?

Our formula diverges as $\omega \to \omega_0$. Classically, the 2p electron should have been radiating at a constant rate since $t = -\infty$, so this is to be expected. Quantum mechanics prevented that until the 1s electron got gobbled up. So the actual distribution will have the peak spread out by uncertainty in the time it takes for the transition.

Real electrons are not scalar charges but have a magnetic dipole moment as well. We could go way back to lecture 20 and add magnetic dipoles to the current source, which we previously took to be entirely due to point charges. But we won't — read Jackson if you want the result.