Lecture 16 March 29, 2010

We know Maxwell's equations and the Lorentz force.
Why more theory?
Newton $\Longrightarrow$ Lagrangian $\Longrightarrow$ Hamiltonian $\Longrightarrow$ Quantum Mechanics
Elegance! — Beauty! — Gauge Fields $\Longrightarrow$ Non-Abelian Gauge Theory $\Longrightarrow$ Standard Model

Anyway, let's look for Lagrangians and actions.

Lagrangian for a Particle

Motion is $\vec{x}(t)$. Action $A = \int L(\vec{x}, \dot{\vec{x}}, t)\, dt$.

Hamilton: actual path extremizes the action.
Doesn't look Lorentz invariant, but all observers must agree (after suitable Lorentz transformation). So A should be a scalar.

Start with a free particle. What could action be?
Can't depend on $\vec{x}$, for translation invariance.
What property of path through space-time can we use?
How about proper length?

$$\begin{aligned} A &= -mc^2 \int d\tau = -mc \int \sqrt{dx^\mu dx_\mu} = -mc \int \sqrt{U^\alpha U_\alpha}\, d\tau \\ &= -mc^2 \int \sqrt{1 - \frac{\vec{u}^2}{c^2}}\, dt. \end{aligned}$$

So the Lagrangian is

$$L(\vec{x}, \vec{u}, t) = -\, mc^2 \big/ \gamma(\vec{u}) = -mc^2 \sqrt{1 - \frac{\vec{u}^{\,2}}{c^2}}.$$

Note L is *not* an invariant, but Ldt and γL are.

Canonical Momentum (in 3-D language)

$$\left(\vec{P}\right)_i = \frac{\partial L}{\partial u_i} = \frac{m u_i}{\sqrt{1 - \frac{\vec{u}^{\,2}}{c^2}}} = (\vec{p})_i,$$

as we previously explored. Euler-Lagrange:

$$\frac{d}{dt}\frac{\partial L}{\partial u_i} - \frac{\partial L}{\partial x_i} = 0,$$

gives $p_i =$ constant, as x_i is an ignorable coordinate. So this is correct for a free particle.

L for particle in a field

What if charge q in an external field?

Can depend on x^μ, but only through the fields' dependence on it. Can involve $U^\alpha = dx^\alpha/d\tau$, but need it in combination as a scalar. Could use A^α or $F^{\alpha\beta}$, but $U_\alpha U_\beta F^{\alpha\beta} \equiv 0$, so only possibility linear in fields is

$$\gamma L_{\text{int}} = -\frac{q}{c} U_\alpha A^\alpha, \quad \Longrightarrow L_{\text{int}} = -q\Phi + \frac{q}{c}\vec{u}\cdot\vec{A},$$

with usual electrostatic and vector potentials. Note first term looks like -PE as expected (as $L = T - V$ often).

So the full lagrangian for the particle is

$$L(\vec{x},\vec{u},t) = -mc^2\sqrt{1-\frac{\vec{u}^{\,2}}{c^2}} + \frac{q}{c}\vec{u}\cdot\vec{A}(\vec{x},t) - q\Phi(\vec{x},t),$$

the canonical momentum becomes

$$\vec{P} = \partial L/\partial\vec{u} = \frac{m\vec{u}}{\sqrt{1-\frac{\vec{u}^{\,2}}{c^2}}} + \frac{q}{c}\vec{A}(\vec{x},t) = \vec{p} + \frac{q}{c}\vec{A},$$

not just the ordinary momentum $\vec{p} = m\gamma\vec{u}$.

The equations of motion are now

$$
\begin{aligned}
\frac{d}{dt}\underbrace{\frac{\partial L}{\partial u_i}}_{P_i}-\frac{\partial L}{\partial x_i} &= \frac{dp_i}{dt}+\frac{q}{c}\underbrace{\frac{d}{dt}\vec{A}_i}_{\left(\frac{\partial A_i}{\partial t}+u_j\partial_j A_i\right)}-\frac{q}{c}u_j\partial_i A_j+q\partial_i\Phi \\
&= \frac{dp_i}{dt}+\frac{q}{c}\frac{\partial \vec{A}_i}{\partial t}+q\partial_i\Phi+\frac{q}{c}\left(u_j\partial_j A_i-u_j\partial_i A_j\right) \\
=0 &= \left(\frac{d\vec{p}}{dt}+\frac{q}{c}\frac{d\vec{A}}{dt}+q\vec{\nabla}\Phi-\frac{q}{c}\vec{u}\times\left(\vec{\nabla}\times\vec{A}\right)\right)_i \\
\frac{d\vec{p}}{dt} &= q\vec{E}+\frac{q}{c}\,\vec{u}\times\vec{B}
\end{aligned}
$$

so we see that this Lagrangian gives us the correct Lorentz force equation.

The Hamiltonian

What is the Hamiltonian? $H = \vec{P}\cdot\vec{u} - L$, but reexpressed in terms of $\vec{P}$ rather than $\vec{u}$. As

$$\vec{u} = \vec{p}/m\gamma(u) = \frac{\vec{p}}{m}\sqrt{1-u^2/c^2} \quad\Longrightarrow\quad \vec{u} = \frac{c\vec{p}}{\sqrt{p^2+m^2c^2}},$$

and $m\gamma(u) = \sqrt{p^2+m^2c^2}/c$. Then we need to substitute $\vec{p} \to \vec{P} - q\vec{A}/c$. Thus

$$\begin{aligned} H &= \frac{\vec{P}\cdot\left(\vec{P}-q\vec{A}/c\right)+m^2c^2}{m\gamma(u)} - \frac{q\left(\vec{P}-q\vec{A}/c\right)\cdot\vec{A}}{cm\gamma(u)} + q\Phi \\ &= \frac{\left(\vec{P}-q\vec{A}/c\right)^2+m^2c^2}{m\gamma(u)} + q\Phi \\ &= \sqrt{(c\vec{P}-q\vec{A})^2+m^2c^4} + q\Phi. \end{aligned}$$

Note H is the total energy, the kinetic energy $p^0c + e\Phi$, so this just verifies $(p^0)^2 - \vec{p}^2 = m^2c^2$.

Adiabatic Invariance of Flux

This L still doesn't have dynamical E&M fields - we will come to that later. First —

Recall from Classical Mechanics: Slowly varying perturbation on an integrable system with cyclic action-angle variables: action is adiabatic invariant. Apply this to motion transverse to uniform static magnetic field.

Action $J = \oint \vec{P}_\perp \cdot d\vec{r}_\perp$ is an invariant.

Need to use *canonical* momentum $\vec{P}_\perp = \vec{p} + q\vec{A}/c$, not just $\vec{p} = m\gamma\vec{v}$. So

$$J = \oint m\gamma\vec{v}_\perp \cdot d\vec{r}_\perp + \frac{q}{c}\oint \vec{A} \cdot d\vec{r}.$$

We have circular motion[1] with $\vec{v}_\perp = -\vec{\omega}_B \times \vec{r}$.

[1] Note J12.38 says $d\vec{v}/dt = \vec{v} \times \vec{\omega}_B = -\vec{\omega}_B \times \vec{v}$, which explains the unexpected minus sign.

So the first term in J is

$$\oint m\gamma\vec{v}_\perp \cdot d\vec{r}_\perp = -\int_0^{2\pi} m\gamma\omega_B a^2 d\theta = -2\pi m\gamma\omega_B a^2.$$

As $m\gamma\vec{\omega}_B = q\vec{B}/c$, this is just $-2q\Phi_B/c$, where Φ_B is the magnetic flux through the orbit.
The second term in J,

$$\frac{q}{c}\oint \vec{A}\cdot d\vec{r} = \frac{q}{c}\int_S \vec{\nabla}\times\vec{A} = \frac{q}{c}\int_S \vec{n}\cdot\vec{B} = \frac{q}{c}\Phi_B,$$

so

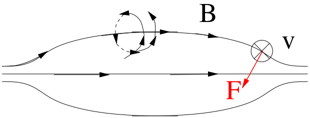
$$-J = q\Phi_B/c = \frac{q}{c}B\pi a^2 = \pi\frac{c}{q}\frac{p_\perp^2}{B}$$

is an adiabatic invariant, as are Ba^2 and $\frac{p_\perp^2}{B}$. These are conserved if $\vec{B}$ varies slowly compared to the gyroradius of the particle's motion.

So $p_\perp^2/B$ may be constant.
In a purely magnetic field, speed and γ are constant, but the transverse speed $v_\perp \propto \sqrt{B}$, while $v^2 = v_\perp^2 + v_\parallel^2$ is constant.

So if particle drifts into a region of stronger B, $v_\perp^2$ may grow to use up all of v^2, and $v_\parallel$ will vanish and reverse. This is a magnetic mirror.

Field lines converge where field gets strong, so Lorentz force has a component pushing particle back into the weaker field region.



This is called a magnetic mirror or magnetic bottle. Note that those particles with negligible $v_\perp$ will not get confined.

treating x^μ as dynamical

The Lagrangian $-mc^2\sqrt{1-\vec{u}^2/c^2}$ certainly doesn't look like a covariant formulation, and we treated it as a functional to determine $\vec{x}(t)$, which is certainly not a covariant way of saying things. On the other hand $-mc\sqrt{dx^\mu dx_\mu} = -mc\sqrt{\eta_{\mu\nu}dx^\mu dx^\nu}$ is a very covariant way of looking at the action, but what do we vary? All of m^μ? or only the spatial part?

Note that if we think of $x^\mu(\lambda)$ as a parameterized path, we may write the action

$$A = -mc\int\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda,$$

and think of varying the function $x^\mu(\lambda)$ and look for an extremum in the usual way. This gives

$$\frac{d}{d\lambda}\left(\frac{\eta_{\mu\nu}\frac{dx^\nu}{d\lambda}}{\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}\right) = 0,$$

or

$$\frac{dx^\mu}{d\lambda} = C^\mu \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}.$$

Doesn't determine $\frac{dx^\mu}{d\lambda}$! Though it looks like four equations, it is really only three, for contracting it with itself gives

$$\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = C^2 \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda},$$

which does nothing to determine $\frac{dx^\nu}{d\lambda}$ but only that $C^2 = 1$.

This should not be surprising. The path length doesn't depend on how it is parameterized, so any change $x^\mu(\lambda) \to x^\mu(\sigma(\lambda))$ will not change A, as long as $\sigma(\lambda)$ is monotone.

Inability to predict the future is a sign of *gauge invariance*, though in this case it is not the gauge invariance we are used to for E&M. Here it is not a serious problem, because we can *choose* to use proper time as our parameter, providing the additional equation

$$\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = c^2, \Longrightarrow \frac{dx^\mu}{d\tau} = \frac{1}{m}p^\mu = \text{ constant.}$$

Action for particles in fields

$$A_{\text{int}} = \frac{-q}{c}\int U^\mu A_\mu \, d\tau = \frac{-q}{c}\int \frac{dx^\mu}{d\tau} A_\mu \, d\tau = \frac{-q}{c}\int A_\tau dx^\mu \quad ?$$

The last expression is clearly covariant, the penultimate one gives the "Lagrangian" for the parameterized path

$$\tilde{L} = -mc\sqrt{\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial\lambda}\frac{\partial x^\beta}{\partial\lambda}} - \frac{q}{c}A_\alpha\frac{\partial x^\alpha}{\partial\lambda}$$

with action $\int \tilde{L} d\lambda$.

$$\begin{aligned} P_\alpha &= -\frac{\partial \tilde{L}}{\partial \frac{\partial x^\alpha}{\partial\lambda}} = \frac{mc\frac{\partial x_\alpha}{\partial\lambda}}{\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} + \frac{q}{c}A_\alpha \\ &\underset{\lambda\to\tau}{\longrightarrow} \quad m\frac{\partial x_\alpha}{\partial\tau} + \frac{q}{c}A_\alpha, \end{aligned}$$

Remember in Euler-Lagrange $d/d\lambda$ is a stream derivative, so

$$\frac{d}{d\tau}A_\alpha = U^\mu\frac{\partial A_\alpha}{\partial x^\mu}.$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{d\tau}P_\alpha &= -\frac{\partial \tilde{L}}{\partial x^\alpha} \\ m\frac{d}{d\tau}U_\alpha + \frac{q}{c}\frac{\partial x^\mu}{\partial \tau}\frac{\partial A_\alpha}{\partial x^\mu} &= +\frac{q}{c}\frac{\partial A_\beta}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tau}, \end{aligned}$$

or

$$m\frac{d}{d\tau}U_\alpha = \frac{q}{c}U^\beta\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right) = \frac{q}{c}F_{\alpha\beta}U^\beta.$$

Canonical Momentum

Canonical Momentum

$$P_\alpha = -\frac{\partial \tilde{L}}{\partial \frac{\partial x^\alpha}{\partial \lambda}} = mU_\alpha + \frac{q}{c}A_\alpha,$$

where we have required our parameter λ to be c times the proper time.
Note that the canonical momentum is constrained:

$$\left(P_\alpha - \frac{q}{c}A_\alpha\right)\left(P^\alpha - \frac{q}{c}A^\alpha\right) = m^2 U_\alpha U^\alpha = m^2 c^2.$$

which we found before as $P^0 = H/c$.
Minimum substitution principle: To introduce electromagnetism for a particle, take a free particle and replace

$$\vec{p}_\alpha \to \vec{P}_\alpha := \vec{p}_\alpha - q\vec{A}/c.$$

Lagrangian for fields

Dynamics of *fields* requires a Lagrangian *density*, a function of the fields[2], say $\phi_i(\vec{x}, t)$. Euler-Lagrange becomes

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial \phi_i / \partial x^\mu)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0.$$

What are our fundamental fields? $\mathcal{L}(\phi_i, \partial_m u \phi_i, x^\nu)$ will give second order differential equations, not Maxwell in F. But we know $\mathbf{F} = d\mathbf{A}$, so second order in A^μ is what we want.

We have already seen particle action requires $-(q/c) A_\mu dx^\mu$ for a single charge. That is, each charge q_i at $\vec{x}_i$ contributes to L $-q_i \Phi(\vec{x}_i) + \frac{q_i}{c} \vec{u}_i \cdot \vec{A}(\vec{x}_i)$.

[2]Never done dynamics of fields? Need to read up, *e.g.* `www.physics.rutgers.edu/~shapiro/507/gettext.shtml` and look at chapter 8 (or get `book9_2.pdf` from the same location).

For many charges,

$$\begin{aligned} L_{\text{int}} &= \sum_i \left(-q_i \Phi(\vec{x}_i) - \frac{1}{c} q_i \vec{u}_i \cdot \vec{A}(\vec{x}_i, t) \right) \\ &\to \int d^3x \left(-\rho(\vec{x})\Phi(\vec{x}) - \frac{1}{c}\vec{J}(\vec{x}) \cdot \vec{A}(\vec{x}) \right) \\ &= -\frac{1}{c} \int d^3x A_\alpha(\vec{x}) J^\alpha(\vec{x}). \end{aligned}$$

This will give us the J_μ on the right hand side of the Euler equation from varying A^μ, but we need something to give the left hand side of Maxwell's equation, which should be linear in F, so we need a quadratic piece in $\mathcal{L}$, Lorentz invariant and with a total of two derivatives on A_μ's. Let's try

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu A^\mu,$$

where it is understood that $F_{\mu\nu}$ stands for $\partial_\mu A_\nu - \partial_\nu A_\mu$ and is not an independent field.

The only contribution to $\partial\mathcal{L}/\partial A_\mu$, (taken with $\partial_\nu A_\mu$ fixed) is the $-J^\mu/c$ from the interaction term. We have

$$\frac{\partial F_{\mu\nu}}{\partial\left(\dfrac{\partial A_\rho}{\partial x^\sigma}\right)} = \delta^\sigma_\mu \delta^\rho_\nu - \delta^\sigma_\nu \delta^\rho_\mu,$$

so

$$\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial A_\rho}{\partial x^\sigma}\right)} = -\frac{1}{4\pi} F_{\rho\sigma},$$

and the full Euler-Lagrange equation is

$$-\frac{1}{4\pi}\partial_\sigma F^{\sigma\mu} + \frac{1}{c} J^\mu = 0,$$

or

$$\partial_\sigma F^{\sigma\mu} = \frac{4\pi}{c} J^\mu.$$

Thus we have derived Maxwell's equations (as $\mathbf{d}F = 0$ is automatic as $\mathbf{F} := d\mathbf{A}$).