Lecture 15 March 25, 2010

We start Chapter 12, *relativistic dynamics of charged particles in interaction with electromagnetic fields.*
But we will do sections 2-4 first, then return to section 1.

In general,
a) electromagnetic fields exert forces on charged particles which alter their motion.
b) the motion of charged particles generates (or alters) electromagnetic fields.
Depending on the situation, one or the other of these may be dominant and we can ignore the other.

For now, we ignore (b), in which case we call $\vec{E}$ and $\vec{B}$ *external* fields.

Motion in External Fields

Many applications can ignore the effect of the charged particles on the fields. These applications include

- bending beams for circular accelerators (nuclear and particle physics)
- plasmas in deep space
- fusion energy devices
- velocity and momentum separators for beams of particles
- the Van Allen belts (auroras)

Begin with $\frac{dp^\alpha}{d\tau} = (q/c)F^{\alpha\beta}U_\beta$, or, in non-relativistic language,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right), \quad \frac{dE}{dt} = q\vec{v} \cdot \vec{E}.$$

Constant Uniform $\vec{B}$, $\vec{E} = 0$

$\vec{B}$ doesn't change energy, so $|\vec{v}|$ and γ are constant,

$$\frac{d\vec{v}}{dt} = \frac{1}{\gamma m}\frac{d\vec{p}}{dt} = \frac{q}{\gamma mc}\vec{v} \times \vec{B} = \vec{v} \times \vec{\omega}_B,$$

where $\vec{\omega}_B = \dfrac{q}{\gamma mc}\vec{B} = \dfrac{qc\vec{B}}{\text{Energy}}$. So component of $\vec{v} \parallel \vec{B}$ is constant, the other two rotate counterclockwise around $\vec{B}$ (for $q > 0$).

Position along $\vec{B}$ grows linearly in time, transverse components of $\vec{r}$ rotate in a circle with angular velocity ω_B. The radius a of this circle is determined from $v_\perp = \omega_B a$, so

$$a = \frac{v_\perp}{\omega_B} = \frac{p_\perp}{m\gamma}\bigg/\frac{qB}{\gamma mc} = \frac{p_\perp c}{qB}.$$

Bending of a beam

This bending of a beam in a magnetic field is used to measure momentum (actually p/q) of particles in beams at all high energy and nuclear accelerators.

What field needed at LHC, 7 TeV protons in a circle with circumference 27 km?
In SI units need extra c: $B = P_\perp/qR$, get $B = 5.4$ T.
Actually need 8.3 T because the magnets don't fill the whole circumference.[1]

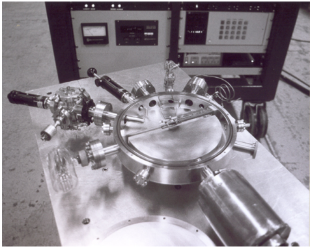
[1] $B = P_\perp c/qR$ in gaussian units, but $B = P_\perp/qR$ in SI units. As $P_\perp \approx E/c$ and $E/q = 7 \times 10^12$ V, $R = 4300$ m, $B = 5.4$ T. Unfortunately the 1232 dipole magnets, each 14.3 m long, do not cover the whole circumference, but only 17.6 km, so the magnets need to be 8.3 T, which is considerably harder to maintain.

That $\omega_B/2\pi = qB/2\pi mc\gamma$, independent of v for $v \ll c$ makes cyclotrons work.

Lawrence	1930	4 in	80 KeV
Lawrence	1931	11 in	1.1 MeV
L&McMillan	1946	184 in	195 MeV*
Koeth	2001	12 in	1 MeV

* synchrocyclotron, deuterons

Lawrence's cyclotron



Koeth's cyclotron

Constant Uniform $\vec{E}$ and $\vec{B}$

With an $\vec{E}$ field, energy no longer constant. But if $\vec{E} \perp \vec{B}$, can use Lorentz transformation to make simpler. Suppose $\vec{E} \parallel y$ and $\vec{B} \parallel z$, and we transform to $\mathcal{O}'$ moving $\parallel x$ with $u_x = c \tanh \zeta$. Then

$$A^{\mu}{}_{\nu} = \begin{pmatrix} \cosh \zeta & \sinh \zeta & 0 & 0 \\ \sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E_y & 0 \\ 0 & 0 & -B_z & 0 \\ E_y & B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow F'^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E'_y & 0 \\ 0 & 0 & -B'_z & 0 \\ E'_y & B'_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned} E'_y &= \cosh \zeta \, E_y - \sinh \zeta \, B_z \\ B'_z &= \cosh \zeta \, B_z - \sinh \zeta \, E_y \end{aligned}$$

More generally, if $\vec{u} \perp \vec{B}$, $\vec{u} \perp \vec{E}$, then

$$\vec{E}' = \gamma(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}), \qquad \vec{B}' = \gamma(\vec{B} - \frac{\vec{u}}{c} \times \vec{E}).$$

Choose $\vec{u} = c\vec{E} \times \vec{B}/B^2$, so

$$\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} + (\vec{E} \times \hat{B}) \times \hat{B}\right) = \gamma\left(\vec{E} - \vec{E} + (\vec{E} \cdot \hat{B})\hat{B}\right) = 0 \\ \vec{B}' &= \gamma\left(\vec{B} - \frac{1}{B^2}(\vec{E} \times \vec{B}) \times \vec{E}\right) = \gamma\vec{B}\left(1 - \frac{E^2}{B^2}\right) = \frac{1}{\gamma}\vec{B}, \end{aligned}$$

$\mathcal{O}'$ sees our previous situation, particle in helix around $\vec{B}'$, but to $\mathcal{O}$ also have an "$\vec{E} \times \vec{B}$ drift" velocity $\vec{u} = c\vec{E} \times \vec{B}/B^2$, in a direction independent of sign of charge, while helical motion reverses with charge.

Important special case: If helical motion degenerates (uniform motion along $\vec{B}'$), $\vec{v}'$ is constant along $\vec{B}'$, but drift is in the $\vec{E} \times \vec{B}$ direction with $u = cE/B$ Only particles with that v_x will travel in a straight line. Apertures then create a velocity selector. You learned all this as freshman, though then you assumed $\vec{u} \perp \vec{B}$.

If $|\vec{E}| > |\vec{B}|$?

If $|\vec{E}| > |\vec{B}|$, the above would give $|\vec{u}| > c$ and imaginary $\vec{B}$, which is not physical. We cannot transform $\vec{E}$ away, but we can transform away $\vec{B}$ instead, with $\vec{u} = c\vec{E} \times \vec{B}/E^2$. Then $\vec{B}\,' = 0$, we have constant uniform $\vec{E}\,'$ and constant $d\vec{p}\,'/dt'$. Nonrelativistically simple ballistic (parabolic) motion, but variation of γ makes solution more difficult, but still doable.

What keeps us from transforming something away? In homework 6 problem 5, you will show $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are invariants. That is why, for $\vec{E} \perp \vec{B}$, there are two different cases, $E^2 - B^2$ negative or positive.

Also, if $\vec{E}$ is **not** perpendicular to $\vec{B}$ in *any one* frame, then $\vec{E} \cdot \vec{B} \neq 0$ in that frame or any other, and they are not perpendicular in any other frame, and neither can be made to vanish. Still the uniform field problem can be solved by brute force.

Constant direction, transverse gradient

Arbitrarily varying fields are not subject to analytic solution, but a useful approximation is perturbation around uniform fields.
In uniform $\vec{B}$, motion is helical. If radius is small compared to scale of variation of $\vec{B}$, perturbation is reasonable.

Consider $\vec{B}(\vec{r})$ parallel to z everywhere and constant in z, so $\vec{B}(\vec{r}) = B(\vec{r}_\perp)\hat{e}_z$, but varying in x on a scale large compared to $\vec{x}_\perp(t) = \vec{r}_\perp(t) - \vec{r}_{0\perp}$, the displacement from the center of the helix of unperturbed motion.
No electric field, so γ constant and v_z constant.
Let $\vec{v}_0(t)$ be the transverse velocity of the unperturbed motion and $\vec{v}_\perp(t) = \vec{v}_0(t) + \vec{v}_1(t)$ be the full transverse velocity. Work to first order in the gradient of B, which is also first order in $\vec{v}_1$.

So $\vec{B} = \hat{e}_z \left(B_0 + \vec{x}_\perp \cdot \vec{\nabla}_\perp B \Big|_0 \right)$, and

$$\frac{d\vec{v}_\perp}{dt} = \frac{q}{\gamma mc} \vec{v}_\perp \times \vec{B}(\vec{x}) \approx \frac{q}{\gamma mc} \vec{v}_\perp \times \vec{B}_0 \left(1 + \frac{1}{B_0} \vec{x}_\perp \cdot \vec{\nabla}_\perp B \Big|_0 \right).$$

$$\begin{aligned} \frac{d(\vec{v}_0 + \vec{v}_1)}{dt} &= \frac{q}{\gamma mc} \vec{v}_0 \times \vec{B}_0 \\ &\quad + \frac{q}{\gamma mc} \vec{v}_1 \times \vec{B}_0 + \frac{q}{\gamma mc} \vec{v}_0 \times \hat{e}_z \left(\vec{x}_\perp \cdot \vec{\nabla}_\perp B \right). \end{aligned}$$

The first term on the right is $d\vec{v}_0/dt$, so

$$\frac{d\vec{v}_1}{dt} = \frac{q}{\gamma mc} \left(\vec{v}_1 \times \vec{B}_0 + \vec{v}_0 \times \hat{e}_z \left(\vec{x}_\perp \cdot \vec{\nabla}_\perp B \right) \right).$$

The unperturbed $\vec{x}_\perp$ is circular motion with radius a, with $\vec{v}_0 \times \hat{e}_z = -\omega_0 \vec{x}_\perp$. So the average $\langle \vec{v}_0 \times \hat{e}_z (\vec{x}_\perp \cdot \vec{\nabla}_\perp B) \rangle = -\omega_0 \langle \vec{x}_\perp (\vec{x}_\perp \cdot \vec{\nabla}_\perp B) \rangle = -\frac{1}{2} \omega_0 a^2 \vec{\nabla}_\perp B$.

We can find a constant drift velocity $\langle \vec{v}_1 \rangle$ on top of the oscillatory motion if $\langle d\vec{v}_1/dt \rangle = 0$. Thus

$$\langle \vec{v}_1 \rangle \times \vec{B}_0 = \frac{1}{2}\omega_0 a^2 \vec{\nabla}_\perp B,$$

or

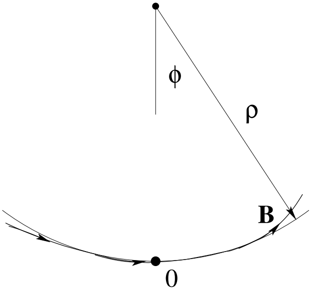
$$\langle \vec{v}_1 \rangle = \frac{1}{B_0^2}\vec{B}_0 \times \left(\langle \vec{v}_1 \rangle \times \vec{B}_0 \right) = \frac{\omega_0 a^2}{2B^2}\vec{B} \times \nabla_\perp B.$$

Thus the particle moves approximately in a helix as before, around a magnetic field line, but the helix drifts in the direction perpendicular to the field line and to the gradient.

Slowly bending $\vec{B}$

At any point, a magnetic field line has a center of curvature. Take that as the origin of polar coordinates, (ρ, ϕ, z), with $\vec{B} = B\hat{e}_\phi$.

Again, $|\vec{v}|$ and γ are constants,



$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{1}{m\gamma}\frac{d\vec{p}}{dt} = \frac{q}{m\gamma}\vec{v}\times\vec{B} = \frac{qB}{m\gamma}\left(v_\rho\hat{e}_z - v_z\hat{e}_\rho\right)\\ &= \frac{qB}{m\gamma}\left(\dot{\rho}\hat{e}_z - \dot{z}\hat{e}_\rho\right)\\ &= \frac{d}{dt}\left(\dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z\right)\\ &= \ddot{\rho}e_\rho + 2\dot{\rho}\dot{\phi}\hat{e}_\phi + \rho\ddot{\phi}\hat{e}_\phi - \rho\dot{\phi}^2\hat{e}_\rho + \ddot{z}\hat{e}_z,\end{aligned}$$

where we have used $d\hat{e}_\rho = \dot{\phi}e_\phi$, $d\hat{e}_\phi = -\dot{\phi}e_\rho$, $d\hat{e}_z = 0$.

$$\frac{qB}{m\gamma}\left(\dot\rho\hat e_z - \dot z\hat e_\rho\right) = \ddot\rho e_\rho + 2\dot\rho\dot\phi\hat e_\phi + \rho\ddot\phi\hat e_\phi - \rho\dot\phi^2\hat e_\rho + \ddot z\hat e_z.$$

The ϕ component gives $2\dot\rho\dot\phi + \rho\ddot\phi = 0$ or $\rho^2\dot\phi = Rv_\parallel$, a constant. The other two components satisfy

$$\ddot\rho - \rho\dot\phi^2 = -\frac{qB}{m\gamma}\dot z, \qquad \ddot z = \frac{qB}{m\gamma}\dot\rho.$$

If[2] $\rho \approx R$, $\dot\rho$ remains bounded, we can ignore $\ddot\rho$ by averaging, we have from the first equation that

$$\langle\dot z\rangle \approx \frac{m\gamma v_\parallel^2}{qBR}.$$

So we have a drift, again in a direction perpendicular to the center of curvature and to the direction of the field.

[2] See lecture notes for how Jackson assures this

Adiabatic Invariants

The last approximation we wish to consider uses the adiabatic invariance of the action. The action involved is $\oint \vec{P}_\perp \cdot d\vec{r}_\perp$ for the motion in the plane perpendicular to the field lines. But before we can discuss this, we need to know the **canonical** momentum $\vec{P}$ conjugate to $\vec{r}$, which **is not** the ordinary momentum $\vec{p} = m\gamma\vec{u}$. To find the **canonical momentum** we need to discuss the Lagrangian.