Lecture 13 March 4, 2010

Algebra of Lorentz Generators

Last time, proper isochronous Lorentz transformations are exponentials of infinitesimal generators,

$$A^\mu{}_\nu = \lim_{N\to\infty}\left[\left(\delta^{\cdot}_{\cdot} + \frac{\omega}{N} L^{\cdot}_{\cdot}\right)^N\right]^\mu{}_\nu = \left(e^{\omega L^{\cdot}_{\cdot}}\right)^\mu{}_\nu,$$

with $L_{\mu\nu} = -L_{\nu\mu}$, real. These are linear combinations of 6 basis matrices, $\mathcal{L}_{\alpha\beta}$, $L^\mu{}_\nu = \sum_{\alpha\beta} c^{\alpha\beta}\left(\mathcal{L}_{\alpha\beta}\right)^\mu{}_\nu$, where $c^{\alpha\beta}$ is antisymmetric, $c^{\alpha\beta} = -c^{\beta\alpha}$, and[1]

$$\left(\mathcal{L}_{\alpha\beta}\right)^{\mu\nu} = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta.$$

Note up-down motion of indices, and that each $\mathcal{L}_{\alpha\beta}$ is a 4×4 matrix, not the $\alpha\beta$ matrix element of one.

[1] The L of Jackson 11.90 is $L^\mu{}_\nu = \left(L_\alpha{}^\beta\right)^\mu{}_\nu$.

$\mathcal{L}_{0j}$ generates a Lorentz boost, $\mathcal{L}_{jk}$ generates a rotation. Define

$$(K_i)^{\cdot}_{\ \cdot} = \left(\mathcal{L}_0{}^i\right)^{\cdot}_{\ \cdot}, \qquad (S_i)^{\cdot}_{\ \cdot} = -\frac{1}{2}\epsilon_{ijk}\left(\mathcal{L}_j{}^k\right)^{\cdot}_{\ \cdot}.$$

The generators S_i are i times the angular momentum operators with a familiar (from QM) commutator algebra, $[S_i, S_j] = \epsilon_{ijk} S_k$. More generally[2]

$$\begin{aligned}
[\mathcal{L}_{\alpha\beta}, \mathcal{L}_{\gamma\zeta}]^{\mu}{}_{\nu} &= \Big(\big\{ \big(\delta^{\mu}_{\alpha}\, \eta_{\beta\rho}\, \delta^{\rho}_{\gamma}\, \eta_{\zeta\nu} - (\alpha \leftrightarrow \beta) \big) - (\gamma \leftrightarrow \zeta) \big\} \\
&\qquad - (\alpha \leftrightarrow \gamma \text{ and } \beta \leftrightarrow \zeta) \Big) \\
&= \eta_{\beta\gamma}\left(\mathcal{L}_{\alpha\zeta}\right)^{\mu}{}_{\nu} - \eta_{\beta\zeta}\left(\mathcal{L}_{\alpha\gamma}\right)^{\mu}{}_{\nu} \\
&\qquad - \eta_{\alpha\gamma}\left(\mathcal{L}_{\beta\zeta}\right)^{\mu}{}_{\nu} + \eta_{\alpha\zeta}\left(\mathcal{L}_{\beta\gamma}\right)^{\mu}{}_{\nu}.
\end{aligned}$$

[2] Lines of algebra skipped — see lecture notes.

$$[\mathcal{L}_{\alpha\beta}, \mathcal{L}_{\gamma\zeta}]^{\mu}{}_{\nu} = \Big(\eta_{\beta\gamma}\left(\mathcal{L}_{\alpha\zeta}\right)^{\mu}{}_{\nu} - (\gamma \leftrightarrow \zeta)\Big) - (\alpha \leftrightarrow \beta).$$

Note: 1) commutator of two generators is a linear superposition of generators.
This closure is one requirement for a Lie Algebra.
Symmetry transformations leaving the invariant length unchanged must form a group, continuous ones make a Lie Group, with generators forming a Lie Algebra.
2) Commutator of a rotation $\mathcal{L}_{ij} = \epsilon_{ijk} S_k$ with a Lorentz boost $\mathcal{L}_{0\ell} = -K_\ell$:

$$\begin{aligned} [S_k, K_\ell] &= -\frac{1}{2}\epsilon_{ijk}\left[\mathcal{L}_{ij}, \mathcal{L}_{0\ell}\right] = \frac{1}{2}\epsilon_{ijk}\left\{\eta_{j\ell}\mathcal{L}_{i0} - \eta_{i\ell}\mathcal{L}_{j0}\right\} \\ &= \epsilon_{k\ell i} K_i, \end{aligned}$$

so the boosts transform like a 3-vector under rotations.
Finally

$$[K_i, K_j] = [\mathcal{L}_{0i}, \mathcal{L}_{0j}] = -\mathcal{L}_{ij} = -\epsilon_{ijk} S_k,$$

so the commutator of two Lorentz boosts is a rotation!

Derivatives

In three dimensions we have the gradient operator $\vec{\nabla}$. In four dimensions we use a different notation,

$$\partial_\mu := \frac{\partial}{\partial x^\mu}.$$

Why covariant? Chain rule:

$$\partial'_\mu = \frac{\partial}{\partial x'^{\,\mu}} = \frac{\partial x^\nu}{\partial x'^{\,\mu}} \frac{\partial}{\partial x^\nu} = A_\mu{}^\nu \partial_\nu,$$

where I have used $A_\mu{}^\nu = \partial x^\nu / \partial x'^{\,\mu}$ derived last time. And that is the way a covariant vector is supposed to transform.

Application to Electromagnetism

We have discussed scalars, invariant under Lorentz transformations, including rotations, and
4-vectors, a combination of a 3-vector and a scalar, and could also have
tensors, with several co- or contra-variant indices.[3]
We saw the energy and momentum are combined into 4-vector $p^\alpha = (E/c, \vec{p})$, but what about the 3-vectors $\vec{E}$ and $\vec{B}$?
From the Lorentz force law[4] for a particle of charge q,

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

while the rate of change of the kinetic energy of the particle is the power provided by the electric field,

$$\frac{dE}{dt} = q\vec{E} \cdot \vec{v}.$$

[3] We will not discuss spinors here.

[4] Jackson has switched to Gaussian units. See lecture notes, or appendix of Jackson.

For a nicely transforming law we should shift to $d/d\tau$, using proper time, and the 4-velocity $U^\alpha = (c\gamma, \gamma\vec{v})$:

$$\begin{aligned}\frac{dp^\alpha}{d\tau} &= \frac{dt}{d\tau}\frac{dp^\alpha}{dt} = q\frac{U^0}{c}\left(\vec{E}\cdot\vec{v}, \vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \\ &= \frac{q}{c}\left(\vec{E}\cdot\vec{U}, U^0\vec{E} + \vec{U}\times\vec{B}\right).\end{aligned}$$

$\vec{E}$ and $\vec{B}$ cannot transform independently — particle at rest ignores $\vec{B}$, one in motion does not. The l.h.s. of this equation is a contravariant 4-vector depending linearly on the 4-vector U^β, but not proportional to it. So the coefficient must be a tensor, ${F^\alpha}_\beta$, with

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c}{F^\alpha}_\beta U^\beta.$$

Matching terms we see

$${F^0}_0 = 0, \quad {F^0}_i = E_i, \quad {F^i}_0 = E_i, \quad {F^i}_j = \epsilon_{ijk}B_k.$$

Field Strength Tensor

If we raise the second index or lower the first, we get

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

$$\begin{aligned} E_i &= F^0{}_i = F^i{}_0 \\ &= -F^{0i} = F^{i0} \\ &= F_{0i} = -F_{i0}. \\ F^i{}_j &= \epsilon_{ijk} B_k \Longrightarrow \\ B_i &= \frac{1}{2}\epsilon_{ijk} F^j{}_k \\ &= -\frac{1}{2}\epsilon_{ijk} F^{jk} \\ &= -\frac{1}{2}\epsilon_{ijk} F_{jk} \end{aligned}$$

Note that F, which is called the *field-strength tensor*, is antisymmetric: $F^{\mu\nu} = -F^{\nu\mu}$.

Forms? consider $\mathbf{F} := \frac{1}{2} F_{\alpha\beta} dx^\alpha dx^\beta$. Then $d\mathbf{F}$ is a 3-form associated with the vector

$$\frac{1}{12} \epsilon^{\alpha\beta\gamma\zeta} \partial_\beta F_{\gamma\zeta},$$

where $\epsilon^{\alpha\beta\gamma\zeta}$ is the totally antisymmetric Levi-Civita symbol for which[5] $\epsilon^{0123} = 1$. The zeroth component of 12 times this is

$$\epsilon^{0\beta\gamma\zeta} \partial_\beta F_{\gamma\zeta} = \epsilon^{ijk} \partial_i F_{jk} = \epsilon^{ijk} \partial_i (-1) \epsilon_{jk\ell} B_\ell = -2 \vec{\nabla} \cdot \vec{B},$$

which vanishes according to one of Maxwell's laws.

[5] In flat space. In general relativity $\epsilon^{0123} = 1/\sqrt{|\det(g_{\mu\nu})|}$.

The i'th spatial component is

$$\begin{aligned}
\epsilon^{i\beta\gamma\zeta}\partial_\beta F_{\gamma\zeta} &= \epsilon^{i0jk}\partial_0 F_{jk} + 2\epsilon^{ijk0}\partial_j F_{k0} \\
&= -\frac{1}{c}\epsilon_{ijk}\left(-\epsilon_{jk\ell}\frac{\partial B_\ell}{\partial t}\right) - 2\epsilon_{ijk}\partial_j(-E_k) \\
&= 2\left(\vec{\nabla}\times\vec{E}\right)_i + \frac{2}{c}\frac{\partial B_i}{\partial t},
\end{aligned}$$

which also vanishes, by another of Maxwell's laws. Thus $d\mathbf{F} = 0$ or

$$\frac{1}{2}\epsilon^{\alpha\beta\gamma\zeta}\partial_\beta F_{\gamma\zeta} = 0 \qquad (1)$$

constitute the sourceless half of Maxwell theory.

Dual Field Strength

This encourages us to also consider the *dual field strength tensor*

$$\mathcal{F}^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\gamma\zeta}F_{\gamma\zeta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix},$$

which is associated with the Hodge dual $*\mathbf{F}$ of the field strength 2-form. Then we can write these Maxwell equations as

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0.$$

The dual field strength tensor is the result of applying the duality of sourceless electromagnetism, $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$.

But Maxwell's equations are not invariant under this duality except in the absence of sources, for there are no magnetic monopoles, at least as far as we currently know. What is the equivalent of $d\mathcal{F}$ or Eq. 1 with $F \to \mathcal{F}$?

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\zeta}\partial_\beta\mathcal{F}_{\gamma\zeta} &= \epsilon^{\alpha\beta\gamma\zeta}\partial_\beta\epsilon_{\gamma\zeta\rho\sigma}\frac{1}{2}F^{\rho\sigma} = 3\left(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho\right)\partial_\beta F^{\rho\sigma} \\ &= 6\partial_\beta F^{\alpha\beta}\end{aligned}$$

Well, the $\alpha = 0$ component of $\partial_\beta F^{\alpha\beta}$ is

$$\partial_j F^{0j} = -\vec{\nabla}\cdot\vec{E} = -4\pi\rho,$$

and the spatial components are

$$\partial_\beta F^{i\beta} = \frac{\partial F^{i0}}{c\partial t} + \partial_j(-\epsilon_{ijk}B_k) = \left(-\vec{\nabla}\times\vec{B} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)_i = -\frac{4\pi}{c}\vec{J}_i.$$

We see that we need to combine $\vec{J}$ and ρ into

$$j^\alpha = \left(c\rho, \vec{J}\right) \quad \text{and we now have} \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu.$$

Notice this has a familiar immediate consequence:

$$\partial_\nu \left(\partial_\mu F^{\mu\nu}\right) = \frac{4\pi}{c}\partial_\nu j^\nu = 0$$

where the vanishing comes because $\partial_\nu\partial_\mu$ is symmetric under $\mu \leftrightarrow \nu$ while $F^{\mu\nu}$ is antisymmetric. Thus we see that

$$\partial_\nu j^\nu = 0 = \frac{\partial c\rho}{c\partial t} + \vec{\nabla}\cdot\vec{J} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J},$$

the equation of continuity follows from Maxwell's equations.

Is ρ like x^0?

Why does the charge density ρ transform like the zeroth component of a 4-vector?
Charge is invariant, so the charge in a given infinitesimal volume, $dq = \rho d^3x$ should be invariant. But volume has a Fitzgerald contraction. In fact, the four-dimensional volume element $d^4x = dx^0 d^3x$ is invariant, because

$$d^4x' = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) d^4x = \det\left(A^\mu{}_\nu\right) d^4x.$$

Taking the determinant of the condition for $A^\cdot{}_\cdot$ to be a Lorentz transformation,

$$\eta_{\alpha\beta} A^\alpha{}_\mu A^\beta{}_\nu = \eta_{\mu\nu} \qquad (2)$$

we have $\det \eta_{\cdot\cdot} (\det A^\cdot{}_\cdot)^2 = \det \eta_{\cdot\cdot}$ or $\det A^\cdot{}_\cdot = \pm 1$.
If we can rule out -1, the answer is yes.

Proper and orthochronous?

Neglected issue: Is any matrix satisfying (2) a Lorentz transformation? Can we have $\det A^{\cdot}{}_{\cdot} = -1$?
If $\mathcal{O}'$'s reference frame was originally boosted from $\mathcal{O}$'s by firing the rocket engines, the velocity relative to $\mathcal{O}$ and the Lorentz transformation should evolve continuously.
As it starts with $A^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}$ which has determinant 1, and a matrix with continuously varying matrix elements has its determinant varying continuously, the determinant cannot jump to -1 and must be 1. So d^4x is invariant and ρ transforms the same way dx^0 and x^0 do.

Are there other constraints on A for a continuous change in v? Taking the 00'th matrix element of (2), we have $\eta_{\mu\nu}A^{\mu}{}_{0}A^{\nu}{}_{0} = (A^{0}{}_{0})^2 - (A^{i}{}_{0})^2 = 1$, so $|A^{0}{}_{0}| \geq 1$. Again, starting at 1, it cannot vary continuously to get to a negative number.

We will call any matrix satisfying (2) a Lorentz transformation, but restrict our attention to those with determinant +1 (proper Lorentz transformations) and with ${A^0}_0 \geq 1$ (orthochronous Lorentz transformations). Note the latter is the condition that time runs in the same direction for both observers. The parity operation $\vec{x} \to -\vec{x}$, t unchanged, is an improper orthochronous Lorentz transformation, while time reversal together with parity, $\vec{x} \to -\vec{x}$, is non-orthochronous but proper. Physics, so far, is invariant under proper orthochronous Lorentz transformations, as Einstein wanted, but not the others (see Wu and Yang, Fitch and Cronin).

Vector Potential

Recall Maxwell tells us $d\mathbf{F} = 0$, $\mathbf{F}$ is closed and should be exact, so what 1-form $\mathbf{A}$ satisfies $\mathbf{F} = d\mathbf{A}$? Or, if you are form-unfriendly, what vector A_ν satisfies

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

Clearly we are to suspect the 4-vector $A^\mu = \left(\Phi, \vec{A}\right)$, where Φ is the electrostatic potential and $\vec{A}$ the usual vector potential. Indeed the $0j$ component of (3) says[6]

$$E_j = \frac{1}{c}\frac{\partial A_j}{\partial t} - \partial_j \Phi = -\left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)_j,$$

and the ij component gives[6]

$$-\epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i = -\epsilon_{ijk}\left(\vec{\nabla} \times \vec{A}\right)_k$$

[6] Note A_μ for $\mu = j$ is $A_j = -A^j = -(\vec{A})_j$. I apologize for this confusing notation, one reason for preferring the opposite choice of sign for η from the one Jackson chooses.

The Lorenz gauge condition is

$$0 = \vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial A^0}{\partial t} = \partial_\mu A^\mu.$$

Finally, the operator for the wave equation in empty space is

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -\partial_\mu \partial^\mu =: -\Box.$$