Lecture 11 March 1, 2010

Last time we mentioned scattering removes power from beam. Today we treat this more generally, to find

- the optical theorem:
- the relationship of the index of refraction and the forward scattering amplitude.

The Optical Theorem

The optical theorem relates the total cross section to the forward scattering amplitude.
In Quantum Mechanics, this is "conservation" of probability. Here — conservation of energy.
We do this differently from Jackson.
Consider scatterer of finite size in an incident plane wave:

$$\begin{aligned}\vec{E}_i &= E_0\,\vec{\epsilon}_i\,e^{i\vec{k}_i\cdot\vec{x}-i\omega t}\\ \vec{B}_i &= \frac{1}{\omega}\vec{k}_i\times\vec{E}_i=\frac{1}{\omega}\vec{k}_i\times\vec{\epsilon}_i\,E_0\,e^{i\vec{k}_i\cdot\vec{x}-i\omega t}\end{aligned}$$

Assume scattering is linear, time-invariant physics, so everything $\propto e^{-i\omega t}$, make implicit.
scattering amplitude $\vec{f}(\vec{k},\vec{k}_i)$, so

$$\begin{aligned}\vec{E}_s(\vec{x}) &= \frac{e^{ikr}}{r}\vec{f}(\vec{k},\vec{k}_i)E_0\\ \vec{B}_s(\vec{x}) &= \frac{1}{\omega}\vec{k}\times\vec{E}_s=\frac{e^{ikr}}{\omega r}E_0\,\vec{k}\times\vec{f}(\vec{k},\vec{k}_i).\end{aligned}$$

Linearity assures frequency unchanged and $k = |\vec{k}| = |\vec{k}_i|$, elastic scattering.

The total fields are $\vec{E} = \vec{E}_i + \vec{E}_s$ and $\vec{B} = \vec{B}_i + \vec{B}_s$.

Consider the power flowing past planes $\perp \vec{k}_i$, one way in front of the scatterer, one way in back.
The total power removed from the incident beam = incident power flux times σ_{tot}, which includes absorption and scattering cross sections.

Take $\vec{k}_i \parallel \hat{z}$. Total power across z downstream is

$$P = \frac{1}{2\mu_0} \int \rho \, d\rho \, d\phi \text{Re} \left[\left(\vec{E}_i + \vec{E}_s \right) \times \left(\vec{B}_i^* + \vec{B}_s^* \right) \right]_z .$$

The $\vec{E}_i \times \vec{B}_i^*$ part of this is what would have been the power without any scattering. Each of $\vec{E}_s$ and $\vec{B}_s^*$ falls off as $1/r$, so the product falls off as $1/r^2$ and is negligible for large r. Thus if ΔP is the change in the power of the beam ($-\delta P$ is the power lost),

$$\Delta P = \frac{1}{2\mu_0} \int \rho \, d\rho \, d\phi \text{Re} \left[\vec{E}_i \times \vec{B}_s^* + \vec{E}_s \times \vec{B}_i^* \right]_z .$$

$$\begin{aligned}\Delta P &= \frac{1}{2\mu_0}\int \rho\, d\rho\, d\phi \mathrm{Re}\left[\vec{E}_i \times \vec{B}_s^* + \vec{E}_s \times \vec{B}_i^*\right]_z \\ &= \frac{1}{2\omega\mu_0}\frac{|E_0|^2}{r}\int \rho\, d\rho\, d\phi \\ &\qquad \mathrm{Re}\left[\vec{\epsilon}_i \times \left(\vec{k} \times \vec{f}^*(\vec{k},\vec{k}_i)\right) e^{-ikr+i\vec{k}_i\cdot\vec{x}}\right. \\ &\qquad \left. + e^{ikr-i\vec{k}_i\cdot\vec{x}}\vec{f}(\vec{k},\vec{k}_i) \times \left(\vec{k}_i \times \vec{\epsilon}_i{}^*\right)\right]_z\end{aligned}$$

For large z, $\rho \sim \sqrt{z}$ so the angle goes to zero, $\vec{k} = \vec{k}_i$, $kr - \vec{k}_i \cdot \vec{x} = k(r-z) = k(\sqrt{z^2+\rho^2} - z) \approx k\rho^2/2z$.

$$\begin{aligned}\int \rho\, d\rho\, d\phi \frac{e^{ikr-i\vec{k}_i\cdot\vec{x}}}{\sqrt{z^2+\rho^2}} &\approx \frac{2\pi}{z}\int_0^\infty \rho\, d\rho\, e^{ik\rho^2/2z} \\ &= \frac{2\pi}{z}\int_0^\infty du\, e^{iku/z} = i\frac{2\pi}{k}.\end{aligned}$$

Thus

$$\begin{aligned}
\Delta P &= \frac{\pi\left|E_0^2\right|}{\omega\mu_0 k}\mathrm{Re}\Big(-i\vec{\epsilon}_i\cdot\vec{f}^*(\vec{k}_i,\vec{k}_i)\vec{k}+i\vec{\epsilon}_i\cdot\vec{k}\vec{f}^*(\vec{k}_i,\vec{k}_i) \\
&\qquad i\vec{\epsilon}_i{}^*\cdot\vec{f}(\vec{k}_i,\vec{k}_i)\vec{k}_i - i(\vec{k}_i\cdot\vec{f}(\vec{k}_i,\vec{k}_i))\vec{\epsilon}_i{}^*\Big)_z \\
&= \frac{\pi\left|E_0^2\right|}{\omega\mu_0}\mathrm{Re}\left(-i\vec{\epsilon}_i\cdot\vec{f}^*(\vec{k},\vec{k}_i)+0+i\vec{\epsilon}_i{}^*\cdot\vec{f}(\vec{k}_i,\vec{k}_i)-0\right) \\
&= -\frac{2\pi\left|E_0^2\right|}{\omega\mu_0}\mathrm{Im}\left(\vec{\epsilon}_i{}^*\cdot\vec{f}(\vec{k}_i,\vec{k}_i)\right).
\end{aligned}$$

The power flux in the incident beam is

$$\frac{1}{2\mu_0}\mathrm{Re}\,(\vec{E}_i\times\vec{B}_i^*)_z = \frac{|E_0|^2}{2\omega\mu_0}\mathrm{Re}\left(\vec{\epsilon}_i\times\left(\vec{k}\times\vec{\epsilon}_i\right)\right)_z = \frac{|E_0|^2 k}{2\omega\mu_0}$$

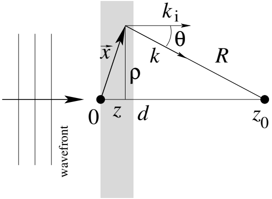
so the total cross section must be

$$\sigma_{\mathrm{Tot}} = \frac{-2\Delta P\omega\mu_0}{|E_0|^2 k} = \frac{4\pi}{k}\mathrm{Im}\left(\vec{\epsilon}_i{}^*\cdot\vec{f}(\vec{k}_i,\vec{k}_i)\right).$$

This is the optical theorem.

Index of Refraction

Consider thin slab, thickness d
Number density N
incident wave $\vec{E}_i = E_0 \vec{\epsilon}_i e^{i\vec{k}_i \cdot \vec{x}}$, $\vec{k}_i = k\hat{e}_z$, observe from far downstream, z_0.
Each d^3x in slab has $N d^3x$ scatterers, so



$$
\begin{aligned}
d\vec{E}_s &= \frac{e^{ikR}}{R} \vec{f}(k, \theta, \phi; \vec{k}_i) E_0 e^{i\vec{k}_i \cdot \vec{x}} N d^3x \\
\vec{E}_s &= N E_0 \int_0^d dz e^{ikz} \int_0^{2\pi} d\phi \int_0^\infty \rho \, d\rho \\
&\qquad \frac{e^{ikR}}{R} \vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)
\end{aligned}
$$

As $R^2 = \rho^2 + (z_0 - z)^2$, $\rho \, d\rho = R \, dR$, so

$$\int_0^\infty \rho\, d\rho\, \frac{e^{ikR}}{R} \vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)$$
$$= \int_{|z_0-z|}^\infty dR\, e^{ikR} \vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)$$
$$= \frac{1}{ik}\, e^{ikR} \vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)\Bigg|_{R=|z_0-z|}^\infty$$
$$-\frac{1}{ik} \int_{|z_0-z|}^\infty e^{ikR}\, dR \frac{d}{dR} \vec{f}\left(k, \cos^{-1}\left(\frac{z_0 - z}{R}\right), \phi; k\hat{e}_z\right)$$

where we integrated by parts for the last expression. The last term is

$$\frac{1}{ik} \int_{|z_0-z|}^\infty e^{ikR}\, dR\, \frac{z_0 - z}{R^2} \frac{d}{d\cos\theta} \vec{f}(k, \theta, \phi; k\hat{e}_z)$$

which, provided the indicated derivative is not singular, falls off like $1/R$.

Dropping that term, we have

$$\begin{aligned}\vec{E}_s &= i\frac{NE_0}{k}\int_0^d dz e^{ikz}\int_0^{2\pi} d\phi e^{ik(z_0-z)}\vec{f}(k,0,\phi;k\hat{e}_z) \\ &= 2\pi i\frac{NE_0 d}{k}e^{ikz_0}\vec{f}(k,0,0;k\hat{e}_z)\end{aligned}$$

Thus the total electric field at points far beyond the slab is

$$\vec{E}(\vec{x}) = E_0 e^{ikz}\left(\vec{\epsilon}_i + \frac{2\pi i N d}{k}\vec{f}(k,0)\right),$$

This is a plane wave, and exact solution of the free space wave equation, though with shifted phase, amplitude, and polarization. Thus it holds right up to back edge of the slab.

What was the effect of the slab? Project on original polarization — initial $\vec{\epsilon}_i{}^* \cdot \vec{E}$ has been multiplied by

$$1 + 2\pi i k^{-1} N \vec{\epsilon}_i{}^* \cdot \vec{f}(k,0)dz.$$

Integrating for finite thickness,

$$\vec{\epsilon}_i^{\,*} \cdot \vec{E}(\vec{x}) = e^{2\pi i k^{-1} N \vec{\epsilon}_i^{\,*} \cdot \vec{f}(k,0) z} E_0 e^{ikz},$$

That is, our wave has k replaced by nk, with

$$n = 1 + \frac{2\pi N \vec{\epsilon}_i^{\,*} \cdot \vec{f}(k,0)}{k^2},$$

which is the index of refraction.
Conclusion: The index of refraction is given by the forward scattering amplitude.

Caveats

We assumed each scatterer feels only the incident field. Better treatment says evaluate $\vec{f}(\vec{k})$ at the wavenumber in the medium, not vacuum.

Absorption will give imaginary part to $k = nk_i = \mathrm{Re}\, k + \frac{i}{2}\alpha k_i$ with

$$\alpha = N\sigma_{\mathrm{tot}} = \frac{4\pi N}{k}\mathrm{Im}\left(\vec{\epsilon}_i^{\,*} \cdot \vec{f}(\vec{k},\vec{k})\right).$$

Only full $\vec{f}$ will satisfy the optical theorem. Approximations suitable for $\mathrm{Im}\, f$ in the forward direction may not work for $\sigma_{\mathrm{tot}} \propto |f^2|$. Example: small lossless dielectric sphere, f is real! So there is scattering but zero total cross section — can't be. In §7.10D Kramers-Kronig said $\mathrm{Re}\,\epsilon_r - 1$ is given by $\int d\omega'$ of $\mathrm{Im}\,\epsilon_r(\omega')$, so a purely real ϵ_r can only be 1.