Ampère's Law — Project 1 Solution

We need to derive the macroscopic form of Ampère's law, including the displacement current.

Start with Ampère's law in the microscopic description,

$$\vec{\nabla} \times \vec{b}(\vec{x},t) - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{e}(\vec{x},t) = \mu_0 \vec{\jmath}(\vec{x},t),$$

where $\vec{b}$ and $\vec{e}$ are the microscopic fields and $\vec{\jmath}(\vec{x},t)$ is the microscopic current density[1]

$$\begin{aligned}\vec{\jmath}(\vec{x},t) &= \sum_{j \text{ free}} q_j \vec{v}_j \delta(\vec{x}-\vec{x}_j) \\ &\quad + \sum_{n \text{ mol}} \sum_{j(n)} q_j \left(\vec{v}_n + \vec{v}_{jn}\right) \delta(\vec{x}-\vec{x}_n-\vec{x}_{jn}).\end{aligned}$$

Here, as in our treatment of the macroscopic version of Gauss' law, q_j are the point charges, divided into the free ones and the ones belonging to molecules. Again n indexes molecules at positions $\vec{x}_n$ (and velocities $\vec{v}_n$) and $\vec{x}_{jn}$ is the relative position of the j'th charge of the n'th molecule from its center, with $\vec{v}_{jn} = \dfrac{d\vec{x}_{jn}}{dt}$.

As the smearing commutes with $\frac{\partial}{\partial t}$, upon smearing the left hand side becomes $\vec{\nabla} \times \vec{B}(\vec{x},t) - \dfrac{1}{c^2}\dfrac{\partial}{\partial t}\vec{E}(\vec{x},t)$ while

$$\begin{aligned}\langle \vec{\jmath}(\vec{x},t)\rangle &= \int d^3x' f(\vec{x}-\vec{x}\,')\Big(\sum_{j \text{ free}} q_j \vec{v}_j \delta(\vec{x}\,'-\vec{x}_j) \\ &\quad + \sum_{n \text{ mol}} \sum_{j(n)} q_j \left(\vec{v}_n + \vec{v}_{jn}\right) \delta(\vec{x}\,'-\vec{x}_n-\vec{x}_{jn})\Big).\end{aligned}$$

The delta functions means we need $f(\vec{x}-\vec{x}_j)$ for the free charges and

$$\begin{aligned}f(\vec{x}-\vec{x}_n-\vec{x}_{jn}) &\approx f(\vec{x}-\vec{x}_n) - \sum_\mu \vec{x}_{jn\,\mu} \frac{\partial}{\partial x_\mu} f(\vec{x}-\vec{x}_n) \\ &\quad + \frac{1}{2}\sum_{\mu\nu} \vec{x}_{jn\,\mu}\, \vec{x}_{jn\,\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} f(\vec{x}-\vec{x}_n) + \ldots\end{aligned}$$

[1] We are assuming the motion is non-relativistic, and ignoring any fundamental term from the spin of elementary particles.

for the bound charges.

Treating $\vec{v}_{jn}$ as the same order as $\vec{x}_{jn}$, the zeroth order term is

$$\vec{J}(\vec{x},t) := \left\langle \sum_{j\,\mathrm{free}} q_j \vec{v}_j \delta(\vec{x}-\vec{x}_j) + \sum_{n\,\mathrm{mol}} \sum_{j(n)} q_j \vec{v}_n \delta(\vec{x}-\vec{x}_n) \right\rangle$$

The first order term is

$$\sum_{n\,\mathrm{mol}} \sum_{j(n)} q_j \left(\frac{\partial \vec{x}_{jn}}{\partial t} f(\vec{x}-\vec{x}_n) - \frac{d\vec{x}_n}{dt} \sum_{\mu} x_{jn\,\mu} \frac{\partial}{\partial x_\mu} f(\vec{x}-\vec{x}_n) \right)$$

Recall $\vec{P}(\vec{x},t) = \sum_{n,j} \vec{q}_j \vec{x}_{jn} f(\vec{x}-\vec{x}_n)$, so the time derivative of the macroscopic polarization is

$$\frac{\partial}{\partial t}\vec{P}(\vec{x},t) = \sum_{n\,\mathrm{mol}} \left(\frac{d\vec{p}_n}{dt} f(\vec{x}-\vec{x}_n) - \vec{p}_n \sum_{\mu} \frac{dx_{n\,\mu}}{dt} \frac{\partial}{\partial x_\mu} f(\vec{x}-\vec{x}_n) \right).$$

With $\vec{p}_n = \sum_{j(n)} q_j \vec{x}_{jn}$, we see that the last two expressions are similar, differing only in which index gets contracted.

So the α'th component of first order term in $\vec{j}(\vec{x},t)$ is

$$\begin{aligned} &\frac{\partial}{\partial t} P_\alpha(\vec{x},t) \\ &+ \sum_{\substack{n\,\mathrm{mol} \\ j(n)}} q_j \sum_{\beta} \left(x_{jn\,\alpha} \frac{dx_{n\,\beta}}{dt} - x_{jn\,\beta} \frac{dx_{n\,\alpha}}{dt} \right) \frac{\partial}{\partial x_\beta} f(\vec{x}-\vec{x}_n). \end{aligned}$$

As $\sum_{j(n)} q_j x_{jn\,\alpha} = p_{n\,\alpha}$, the second row here is

$$\sum_{\beta} \frac{\partial}{\partial x_\beta} \left\langle \sum_{n} \left(p_{n\,\alpha} v_{n\,\beta} - p_{n\,\beta} v_{n\,\alpha} \right) \delta(\vec{x}-\vec{x}_n) \right\rangle. \tag{1}$$

This agrees with the second line of Jackson 6.96

Jackson claims we should define a *molecular magnetic moment*

$$\vec{m}_n = \sum_{j(n)} \frac{q_j}{2} \left(\vec{x}_{jn} \times \vec{v}_{jn} \right)$$

and the *macroscopic magnetization*

$$\vec{M}(\vec{x},t) = \left\langle \sum_{n\,\mathrm{mol}} \vec{m}_n \delta(\vec{x}-\vec{x}_n) \right\rangle.$$

His equation for $\langle j_\alpha(\vec{x},t)\rangle$ contains a term

$$\sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\beta} M_\gamma(\vec{x},t) =$$

$$\sum_{\beta\gamma\mu\nu} \epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu} \sum_{n\,\mathrm{mol}} \sum_{j(n)} \frac{q_j}{2} x_{jn\,\mu} v_{jn\,\nu} \frac{\partial}{\partial x_\beta} f(\vec{x}-\vec{x}_n)$$

From the definition of $\vec{D}$ (Jackson 6.92), there is one more term in $\partial(D-\epsilon_0 E)/\partial t$, which is

$$-\frac{\partial}{\partial t} \sum_\beta \frac{\partial}{\partial x_\beta} \mathbf{Q}_{\alpha\beta} = -\frac{1}{2} \sum_{n,j,\beta} q_j \frac{d}{dt} \left(x_{jn\,\alpha} x_{jn\,\beta}\right) \frac{\partial}{\partial x_\beta} f(\vec{x}-\vec{x}_n)$$

$$+ \frac{1}{2} \sum_{n,\beta} \left(\sum_{j(n)} q_j x_{jn\,\alpha} x_{jn\,\beta} \right) \sum_\gamma v_{n\,\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} f(\vec{x}-\vec{x}_n).$$

The first line here and the magnetization expression both involve $\displaystyle\sum_{n,j} \frac{q_j}{2}\, x_{jn\,\mu}\, v_{jn\,\nu} \frac{\partial}{\partial x_\beta} f(\vec{x}-\vec{x}_n)$, the magnetization term contracted into $\sum_\gamma \epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$, and the $D-\epsilon E$ term contracted into $-\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$, so the two add to

$$-\sum_{n,j} q_j\, x_{jn\,\beta}\, v_{jn\,\alpha} \frac{\partial}{\partial x_\beta} f(\vec{x}-\vec{x}_n).$$

The second line is

$$\frac{1}{6} \sum_n \sum_{\beta\gamma} v_{n\,\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \left\langle \mathbf{Q}_{n\,\alpha\beta} \delta(\vec{x}-\vec{x}_n) \right\rangle$$

All together,

$$\left(\vec{\nabla}\times\vec{M}\right)_\alpha - \frac{\partial}{\partial t} \sum_\beta \frac{\partial}{\partial x_\beta} \mathbf{Q}_{\alpha\beta}$$

$$= -\sum_{n,j,\beta} q_j\, x_{jn\,\beta}\, v_{jn\,\alpha} \frac{\partial}{\partial x_\beta} f(\vec{x}-\vec{x}_n)$$

$$+ \frac{1}{6} \sum_n \sum_{\beta\gamma} v_{n\,\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \left\langle \mathbf{Q}_{n\,\alpha\beta} \delta(\vec{x}-\vec{x}_n) \right\rangle.$$

How does this compare to what we need for $\langle j_\alpha(\vec{x},t)\rangle$?

The second order (in x_{jn} and v_{jn}) terms in $\langle j_\alpha(\vec{x},t)\rangle$ are

$$\sum_{\substack{n \text{ mol} \\ j(n)}} q_j \left(v_{n\,\alpha} x_{nj\,\beta} x_{nj\,\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} - v_{nj\,\alpha} x_{nj\,\beta} \frac{\partial}{\partial x_\beta} \right) f(\vec{x}-\vec{x}_n)$$
$$= \frac{1}{6} \sum_n v_{n\,\alpha} \sum_{\beta\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \langle \mathbf{Q}_{n\,\beta\gamma} \delta(\vec{x}-\vec{x}_n)\rangle + \left(\vec{\nabla}\times\vec{M}\right)_\alpha$$
$$-\frac{\partial}{\partial t}\frac{\partial}{\partial x_\beta}\mathbf{Q}_{\alpha\beta} - \frac{1}{6}\sum_n v_{n\,\beta\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \langle \mathbf{Q}_{n\,\alpha\beta} \delta(\vec{x}-\vec{x}_n)\rangle\,.$$

This verifies Jackson's equation 6.96. As the expression averaged in (1) is antisymmetric in $\alpha \leftrightarrow \beta$, we can write (1) as

$$\sum \epsilon_{\alpha\mu\gamma} \frac{\partial}{\partial x_\beta} \epsilon_{\gamma\mu\nu} \left\langle \sum_n p_{n\,\mu} v_{n\,\nu}\, \delta(\vec{x}-\vec{x}_n) \right\rangle$$
$$= \left(\vec{\nabla} \times \left\langle \sum_n \vec{p}_n \times \vec{v}_n\, \delta(\vec{x}-\vec{x}_n) \right\rangle \right)_\alpha .$$

The same applies to the quadripole terms, where we have

$$-\frac{1}{6} \sum_{\beta\gamma\mu\rho} \epsilon_{\rho\alpha\gamma} \epsilon_{\rho\mu\nu} \sum_n v_{n\,\mu} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \langle \mathbf{Q}_{n\,\nu\beta}\, \delta(\vec{x}-\vec{x}_n)\rangle$$
$$= \left(\vec{\nabla} \times \sum_n \left(\vec{v}_n \times \sum_\nu \hat{e}_\nu \sum_\beta \frac{\partial}{\partial x_\beta} \langle \mathbf{Q}_{n\,\nu\beta}\, \delta(\vec{x}-\vec{x}_n)\rangle \right) \right)_\alpha .$$

$\vec{H}$ at last

Let us define the *magnetic field* as

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} - \left\langle \sum_n \vec{p}_n \times \vec{v}_n\, \delta(\vec{x}-\vec{x}_n) \right\rangle$$
$$-\frac{1}{6}\sum_n \left(\vec{v}_n \times \sum_\nu \hat{e}_\nu \sum_\beta \frac{\partial}{\partial x_\beta} \langle \mathbf{Q}_{n\,\nu\beta}\, \delta(\vec{x}-\vec{x}_n)\rangle \right).$$

Then Ampère's microscopic law smeared, $\vec{\nabla}\times\vec{B} - (1/c^2)\partial\vec{E}/\partial t = \mu_0 \langle \jmath(\vec{x},t)\rangle$ gives us Ampère + Maxwell in media:

$$\vec{\nabla}\times\vec{H} - \frac{1}{c^2}\frac{\partial \vec{D}}{\partial t} = \vec{J}(\vec{x},t).$$

$$\text{with} \quad \vec{J}(\vec{x},t) := \left\langle \sum_{j\,\text{free}} q_j \vec{v}_j \delta(\vec{x}-\vec{x}_j) + \sum_{n\,\text{mol}} \sum_{j(n)} q_j \vec{v}_n \delta(\vec{x}-\vec{x}_n). \right\rangle$$

For most purposes, we may drop the terms other than $\vec{M}$ in the difference between $\vec{B}$ and $\mu_0 \vec{H}$.