Intro to Lecture 16 Nov. 2, 2016

Last time, after reminding ourselves of how some important second order linear ordinary differential equations emerge, we began our analysis of how to solve these equations and more general ones. We defined ordinary, regular singular and essential singular points and looked for expansions of the solutions about that point. This gave us Frobenius' method for finding at least one solution expanding about an ordinary or regular singular point, by setting

$$y(x) = (x - x_0)^k \sum_{n=0}^{\infty} a_n \,(x - x_0)^n$$

with k the larger solution of the indicial equation $\big[k(k-1) + b_0 k + c_0\big] = 0$, and using the recursion relation

$$\left[(p+k)(p+k-1) + b_0(p+k) + c_0\right] a_p = -\sum_{n=0}^{p-1} a_n \left(b_{p-n}(n+k) + c_{p-n}\right)$$

Then we introduced the Wronskian which enables us to find the other solution if the smaller solution of the indicial equation doesn't work.

Today we will apply this to Bessel's equation and find the Neumann functions.

Then we will turn to a more general formulation of the problem of the space of solutions on an open interval (a, b) with no singular points, probably bounded by singular points. Because we generally have a separation constant which can take on many values (generally a discrete but infinite set of them), we may consider the vector space of all the homogeneous solutions, and place a metric, or more importantly an inner product, given by a an integral. Initially we will consider the inner product $(v, u) := \int v^*(x) u(x)\, dx$, and consider the second order differential equation as a second order differential operator acting on the solutions. We will rephrase our equation by multiplying by a function of x to make the operator hermitean. But the differential equation that emerges is not quite an eigenfunction equation, which would be nice. So we redefine our inner product to have a positive weight function $w(x)$, $\langle v, u\rangle := \int w(x)\, v^*(x) u(x)\, dx$. Choosing the weight function appropriately from the function multiplying the separation constant, we will find the our solutions form an orthonormal basis of an infinite dimensional vector space. Then we can use the techniques and knowledge we have from linear algebra, albeit with some caution as we are now in an infinite-dimensional vector space.

- Homework #7 is due Monday at 5:00 PM as usual.
