Intro to Lecture 6 Sept 23, 2016

Last time we discussed the idea that a manifold should be described by an atlas of charts, each a 1-1 map of some open set of the manifold into an open set in $\mathbb{R}^n$, with the intrinsic infinitesimal distances on the manifold given on a chart C_k by the covariant metric tensor

$$(ds)^2 = \sum_{j\ell} g_{k\ell}\, dq^j dq^\ell,$$

where $\{dq^j\}$ is a basis of 1-forms in $\mathcal{T}^*_{\mathcal{P}}$, which is called the cotangent space at the point $\mathcal{P}$. We found that geodesics, the shortest paths between two points, are described by

$$\ddot{q}^j + \Gamma^j{}_{mn}\dot{q}^m\dot{q}^n, \qquad \text{where} \quad \dot{q}^j = \frac{dq^j}{ds},$$

and $\Gamma^j{}_{mn}$ is the Christoffel symbol given by the metric and its derivatives.

The differential of a scalar field $f : \mathcal{M} \to \mathbb{R}$ is defined by $\mathbf{d}f$ and represented in chart C_k by

$$\mathbf{d}f = \sum_j \frac{\partial \tilde{f}}{\partial q^j}\, dq^j,$$

where $\tilde{f} = f \circ \phi_k^{-1} : C_k \to \mathbb{R}$. The differential was our first example of a 1-form, but we defined a more general 1-form on $\mathcal{M}$ by

$$\boldsymbol{\omega}(\mathcal{P}) = \sum_j \omega_j(\phi_k(\mathcal{P}))dq^j$$

where the *covariant* vector field ω_j transforms under change of chart $C \to C'$ as

$$\omega'_j = \frac{\partial q^k}{\partial q'^j}\omega_k.$$

We also found some physical objects are better defined in the dual space, $\mathcal{T}_{\mathcal{P}}$ by *contravariant vectors*, so the electric field E_k in chart C is transformed so that $Q\,\mathbf{E}\,d\mathcal{P}$, the work done, is intrinsic on the manifold, independent of the choice of chart used to describe $\mathbf{E}$ and $d\mathcal{P}$.