Physics 464/511 Lecture O Fall, 2016

1 Relativity

I will begin our discussion of general relativity and our further exploration of differential geometry with a brief review of special relativity. First of all, we are looking at the dynamics of particles and fields in three dimensional space and in time, so the manifold we need to look at is a four dimensional manifold which, to begin with, we can consider a Minkowski space with cartesian coordinates x^μ. Einstein's insight was to take seriously the notion that the laws of physics should be the same for inertial reference frames. That is, if we have the natural chart of an observer $\mathcal{O}'$ who is moving with constant velocity $\vec{V}$ with respect to observer $\mathcal{O}$, the laws of physics stated in $\mathcal{O}'$'s coordinates x'^μ are the same as for $\mathcal{O}$.

Of course Galileo said that as well. And if we look at Newton's law for $\mathcal{O}'$, $\vec{F}' = md^2\vec{x}'/dt'^2$ and use the Galilean transition function between their charts: $\vec{x}' = \vec{x} - \vec{V}t,\ t' = t$, we have

$$\vec{F}' = m\frac{d^2}{dt^2}(\vec{x} - \vec{V}t) = m\frac{d^2}{dt^2}\vec{x} = \vec{F}$$

so without change of force, Newton's law is invariant. [Note: $\vec{V}$ needed to be constant, which defines $\mathcal{O}'$ as an inertial frame.]

But Einstein looked at a different law of physics — that the speed of light is a given constant c independent of the source. This is not invariant under the Galilean transition function, which would say that a beam of light traveling at c in the direction of $\vec{V}$ with respect to $\mathcal{O}$ has velocity $c - V$ with respect to $\mathcal{O}'$. Rather, Relativity tells us that, looking at two events connected by a light ray (in vacuum, of course), say $\mathcal{P}_1 = (t_1, \vec{r}_1)$ and $\mathcal{P}_2 = (t_2, \vec{r}_2)$, we must have both $|\vec{r}_2 - \vec{r}_1| = c(t_2 - t_1)$ and $|\vec{r}\,'_2 - \vec{r}\,'_1| = c(t'_2 - t'_1)$. Defining an indefinite metric $ds^2 = (d\vec{r}\,)^2 - c^2dt^2$ we see that the requirement is that all observers must agree that if one finds $ds^2 = 0$, so must the others. Electing to define $x^0 = ct$ and the $x^j,\ j = 1, 2, 3$ the components of $\vec{r}$, we have $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$, summed of course on μ and ν from 0 to 3, with $\eta_{\mu\nu}$ the diagonal matrix[1] with $\eta_{00} = -1$, $\eta_{jk} = \delta_{jk}$

[1]Greek indices are summed from 0 to 3, latin indices from 1 to 3.

Also, as particles without forces on them travel in straight lines in space-time, the transition function $x'^\mu(x^\nu)$ must be linear, and these two requirements mean we must have

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{1}$$

with the condition[2]

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{2}$$

The chart transition (1) subject to this Lorentz tranformation condition is called a Poincaré transformation.

In special relativity the metric tensor $g_{\mu\nu}$ is fixed to be the constant matrix $\eta_{\mu\nu}$, though when we get to general relativity this will change, with $g_{\mu\nu}$ becoming a dynamic degree of freedom. But for the time being the difference between contravariant vectors v^ν and covariant vectors v_ν is just the sign of the time component. The dot product is defined with the metric, so $\mathbf{V}\cdot\mathbf{W} = \eta_{\mu\nu}V^\mu W^\nu = -V^0W^0 + \sum_j V^jW^j$. The nature of the momentum 4-vector is most easily found from quantum mechanics, where $\vec{p} = -i\hbar\vec{\nabla}$, $E = i\hbar\partial/\partial t$, so with $P_\mu = -i\hbar\partial/\partial x^\mu$, we have $E = -cP_0 = cP^0$, $\vec{p} = P^j\hat{e}_j$. Then the square of the 4-momentum is $P^\mu P_\mu = -E^2/c^2 + \vec{p}\,^2 = -m^2c^2$, the same for all observers.

Suppose we have a set of particles with electric charge q_n and trajectories $\vec{x}_n(t)$. At any given time t, the charge density is clearly

$$\rho(\vec{x}, t) = \sum_n q_n\delta^3(\vec{x} - \vec{x}_n(t)).$$

Current is a rate of flow of charge past a given plane, and can be seen to be the density times velocity for a uniformly moving body, in an argument you have probably seen several times before in E&M or thermal. Thus

$$\vec{J}(\vec{x}, t) = \sum_n q_n\delta^3(\vec{x} - \vec{x}_n(t))\vec{v}_n.$$

To make four dimensional, let $x^\mu_n(\lambda)$ be the world line in terms of an arbitrary parameter λ. A particle whose position as a function of time is

[2]Actually this does not follow from what we have required sofar, only that if $(ds)^2 = 0$ we must have $(ds')^2 = 0$, while (2) requires $(ds)^2 = (ds')^2 = 0$ for all space-time intervals. This does follow, however, from the requirement that all observers agree on the mean lifetimes of particles, or their masses.

given by $\vec{x}(t)$ (for some observer $\mathcal{O}$) traces a path through 4-dimensional space along an arbitrarily parameterized path $x^\mu(\lambda)$ with $x^0(\lambda) = ct$, so $x^j(\lambda) = \vec{x}(x^0(\lambda)/c)$, provided we have a parameterization for which $x^0(\lambda)$ is monotone. Of course things look more transparent if we choose $\lambda = t$. In 4-dimensional terms we define

$$J^\mu(x^\nu) = \int d\lambda \sum_n q_n \delta^4(x^\nu - x_n^\nu(\lambda)) \frac{dx^\mu}{d\lambda}.$$

If $\lambda = t$ for each world line, clearly this reduces to the previous definitions. Furthermore the definition is independent of the parameterization, for if $\tilde{x}(\tilde{\lambda}) = x(\lambda)$,

$$\int d\tilde{\lambda}\, \delta^4(x^\nu - \tilde{x}_n^\nu(\tilde{\lambda})) = \int d\lambda\, \delta^4(x^\nu - x_n^\nu(\lambda)) \frac{d\tilde{\lambda}}{d\lambda}.$$

The condition on Λ for a Lorentz transformation, $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ as matrices reads $\Lambda^T \eta \Lambda = \eta$, so taking the determinant tells us[3] $\det \Lambda^{\cdot}{}_{\cdot} = \pm 1$, the Jacobian has absolute value 1, and therefore the dirac delta δ^4 is invariant, so J^μ is a contravariant vector.

In nonrelativistic physics, we learn the conservation equation in the form $\vec{\nabla}\cdot\vec{J} + \dfrac{\partial \rho}{\partial t} = 0$, or $\displaystyle\sum_\mu \frac{\partial J^\mu}{\partial x^\mu} = 0$. To verify that,

$$\frac{\partial}{\partial x^\nu} J^\mu(x^\nu) = \int d\lambda \sum_n q_n \left[\frac{\partial}{\partial x^\nu}\delta^4(x - x_n(\lambda))\right] \frac{dx_n^\mu}{d\lambda}.$$

$$\begin{aligned}\text{Now } \frac{d}{d\lambda}\delta^4(x - x_n(\lambda)) &= \frac{dx_n^\mu(\lambda)}{d\lambda}\frac{\partial}{\partial x_n^\mu}\delta^4(x - x_n(\lambda)) \\ &= -\frac{dx_n^\mu(\lambda)}{d\lambda}\frac{\partial}{\partial x^\mu}\delta^4(x - x_n(\lambda))\end{aligned}$$

so

$$\begin{aligned}\frac{\partial}{\partial x^\mu} J^\mu(x) &= \int d\lambda \frac{d}{d\lambda}\left(\sum_n q_n \delta^4(x - x_n(\lambda))\right) \\ &= \sum_n q_n\, \delta^4(x - x_n(\lambda))\Big|_{\lambda=-\infty}^{\lambda=+\infty} = 0\end{aligned}$$

[3]We will generally be interested only in proper isochronous transformations, for which $\det \Lambda^{\cdot}{}_{\cdot} = 1$.

if we assume that particles start in the infinite past and end in the infinite future, neither of which is x^0, which is now.

In general conserved quantities are equivalent to a divergenceless 4-current, which is therefore often called a conserved current. The total charge for such a quantity

$$Q(t) = \int d^3x\big|_{t=\text{constant}} J^0(\vec{x}, t) \quad \text{satisfies}$$

$$\frac{dQ}{dt} = \int d^3x \frac{\partial}{\partial x^0} J^0 = -\int d^3x\, \vec{\nabla}\cdot\vec{J} = -\int dS\, \hat{n}\cdot\vec{J}$$

where S is a surface (at infinity) surrounding the volume over which we are calculating the charge. If it is the total charge, the volume is all of space and the surface is at infinity. We may assume, usually, that all physical events are happening with some bounded region, (at least events which affect our experiments) so we may assume $\vec{J}$ is zero as we go infinitely far away, and then $\frac{dQ}{dt} = 0$, or Q doesn't change (is conserved).

We saw in lecture 8 or E that the electromagnetic fields are determined by derivatives of the 4-vector field $A_\mu(x)$ into the field-strength tensor $F^{\mu\nu}$, which in terms of n-forms is $\boldsymbol{F} = d\boldsymbol{A}$, and which implies two of Maxwell's equations $d\boldsymbol{F} = 0$. We also found $d * \boldsymbol{F} = *\boldsymbol{J}$, which are the other two equations. The only remaining piece of electromagnetism is the Lorentz force law, which we used to think of as

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right),$$

but we should really think of as

$$f^\mu = \frac{dP^\mu}{d\tau} = qF^{\mu\nu}u_\nu,$$

where u^ν is the 4-velocity $dx^\mu/d\tau$ and τ is the proper time $\tau = \frac{1}{c}\int\sqrt{-(ds^2)}$, and f^μ is the 4-force, which is only the same as the ordinary force if $d\tau = dt$, which is to say, in the rest frame of the particle in question.

Notice that $u^2 = \eta_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = \dfrac{(ds)^2}{(d\tau)^2} = -1$, so we see that

$$P^\mu = m\frac{dx^\mu}{d\tau} = m\, u^\mu.$$

2 Stress-Energy Tensor

Recall that if a "charge" q_n is associated with each particle n, we may define a current

$$J^\nu(x) = \int d\lambda \sum_n q_n \delta^4\left(x - x_n(\lambda)\right) \frac{dx_n^\nu(\lambda)}{d\lambda}.$$

such currents may be written for any property carried by the particles, not just the electric charge. In particular, each particle has momentum p^μ, so we may write

$$T^{\mu\nu}(x) = \int d\lambda \sum_n p_n^\mu \delta^4\left(x - x_n(\lambda)\right) \frac{dx_n^\nu(\lambda)}{d\lambda}.$$

This object is called the stress-energy tensor. It is independent of the choice of parameter λ. Two special choices are

1. $\lambda = t$, $T^{\mu\nu}(\vec{x}, t) = \sum_n p_n^\mu \delta^3\left(\vec{x} - \vec{x}_n(t)\right) \frac{dx^\nu}{dt}$ as $\int dt' \delta(t-t') = 1$. Thus $T^{\mu j}$ is the flux of momentum p^μ across a surface perpendicular to the j direction, just as $\vec{J}$ is the current per unit area across a boundary. The components $T^{\mu 0}$ are the density of the μ component of momentum.

2. $\lambda = \tau$, $T^{\mu\nu}(x) = \displaystyle\int d\tau \sum_n \delta^4(x - x_n(\tau)) m_n \frac{dx_n^\mu}{d\tau}\frac{dx_n^\nu}{d\tau}$. In this form we see that $T^{\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$. We also see that it is a tensor, transforming like $dx^\mu \otimes dx^\nu$.

Conservation:

$$\begin{aligned}
\partial_\nu T^{\mu\nu}(x) &= \sum_n \int d\lambda\, p_n^\mu(\lambda) \frac{dx_n^\nu}{d\lambda} \underbrace{\underbrace{\frac{\partial}{\partial x^\nu}\delta^4\left(x - x_n(\lambda)\right)}_{-\frac{\partial}{\partial x_n^\nu}\delta^4\left(x - x_n(\lambda)\right)}}_{-\frac{d}{d\lambda}\delta^4\left(x - x_n(\lambda)\right)} \\
&= -\sum_n p_n^\mu(\lambda)\, \delta^4\left(x - x_n(\lambda)\right)\Big|_{\lambda=-\infty}^{\lambda=+\infty} + \sum_n \int d\lambda \delta^4\left(x - x_n(\lambda)\right) \frac{dp_n^\mu}{d\lambda}.
\end{aligned}$$

The first term is zero for any finite x assuming the particles go off to infinity, at least for x^0, as $\lambda \to \pm\infty$. In the second term we can take $\lambda = t$, so it

reduces to $\displaystyle\sum_n \delta^3\left(\vec{x} - \vec{x}_n(t)\right) \underbrace{\frac{dp_n^\mu}{dt}}_{\frac{d\tau}{dt} f_n^\mu}$. Thus

$$\partial_\nu T^{\mu\nu}(x) = G(x) = \sum_n \delta^3\left(\vec{x} - \vec{x}_n(t)\right) \frac{d\tau_n}{dt} f_n^\mu.$$

If the particles are free, $f = 0$. Even if they interact at a point,

$$\partial_\nu T^{\mu\nu}(x) \simeq \sum_{x=x_n} \underbrace{\frac{d\tau_n}{dt} f_n^\mu}_{F^\mu = \frac{dP^\mu}{dt}} = \frac{d}{dt} \sum_{\substack{\text{particles}\\ \text{involved}}} P_n^\mu.$$

We expect the total momentum of the colliding particles to be conserved, so $\frac{d}{dt}\sum P_n^\mu = 0$, and

$$\partial_\nu T^{\mu\nu}(x) = 0.$$

When is it not zero?

1. If there's is an external field influencing p_n

2. if the particles interact at a distance.

Action at a distance would not conserve $T^{\mu\nu}$ because momentum is then transferred out of a region without any physical flow of momentum through the walls of the region. While this is allowed by Newton's laws and required by his formulation of gravity (the forces act instantanteously) this notion violates relativity. Consider two masses at rest. Move #1 up. Newton's law of gravity, or Coulomb's law, would tell you that particle #2 immediately feels a change in the direction of the force, hence carrying a signal faster than light can travel. We know that this is not true. In electromagnetism, other forces, due to the moving charges and radiating fields, cancel the effect of the change from Coulomb's law. In fact, we know it is better to think of one charge as producing the field, changes in which can propagate only at the velocity of light, and the other charge sensing the force locally through the field.

We will assume there are no actions at a distance mechanisms in physics, and all forces apparently such are in fact conveyed by a field. We have so far discussed the energy momentum only of the particles, not including the energy and momentum of the field.

To see how to add this in, consider electromagnetism,

$$\begin{aligned}\partial_\nu T^{\mu\nu}_{\text{particles}}(x) &= \sum_{x=x_n} \delta^3\left(\vec{x}-\vec{x}_n(t)\right)\frac{d\tau_n}{dt}\left(f^\mu_n = q_n F^{\mu\rho}(x_n)\frac{dx_{n\,\rho}}{d\tau_n}\right)\\ &= \sum_n q_n\delta^3\left(\vec{x}-\vec{x}_n(t)\right)F^\mu{}_\rho(x_n)\frac{dx^\rho_n}{dt} = F^\mu{}_\rho(x)J^\rho(x).\end{aligned}$$

[Note the order of indices is important, $F^\mu{}_\rho \neq F_\rho{}^\mu$.]

What should the stress-energy tensor of the electromagnetic field itself be? The energy density is[4]

$$T^{00} = \frac{1}{2}\left(E^2+B^2\right) = \frac{1}{2}F^{0i}F^{0i} + \frac{1}{4}F^{ij}F^{ij},$$

and the energy flux is

$$T^{0i} = S^i = \left(\vec{E}\times\vec{B}\right)^i = F^{0j}F^i_{\ j}.$$

This hints that T should be quadratic in F, and depend on nothing else (except, of course, the constant matrices η and ϵ. Considering Lorentz covariance and symmetries, the only possibilities are

$$T^{\mu\nu} = aF^{\mu\rho}F^\nu{}_\rho + b\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma},$$

but then $T^{00} = aF^{0i}F^{0i} + 2bF^{0i}F^{0i} - bF^{ij}F_{ij}$, so we must have $b=-1/4$, $a+2b=1/2$, so $a=1$,

$$T^{\mu\nu}_{\text{Maxwell}} = F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}.$$

We see that

$$\begin{aligned}\partial_\nu T^{\mu\nu}_{\text{Maxwell}} &= \left(\partial_\nu F^{\mu\rho}\right)F^\nu{}_\rho - F^{\mu\rho}J_\rho - \frac{1}{2}\eta^{\mu\nu}F^{\rho\sigma}\partial_\nu F_{\rho\sigma}\\ &= -F^{\mu\rho}J_\rho + F_{\alpha\beta}\left[\partial^\alpha F^{\mu\beta} - \frac{1}{2}\partial^\mu F^{\alpha\beta}\right].\end{aligned}$$

[4] We are using units with $\mu_0 = \epsilon_0 = 1$ and therefore $c=1$. The energy density and the Poynting vector giving the energy flux should be familiar (*e.g.* Young and Freedman, §32.4).

Note that only the part of the bracket antisymmetric under $\alpha\leftrightarrow\beta$ survives contracting with $F_{\alpha\beta}$, so

$$[\,]\to\frac{1}{2}\partial^\alpha F^{\mu\beta} - \frac{1}{2}\partial^\beta F^{\mu\alpha} - \frac{1}{2}\partial^\mu F^{\alpha\beta} = -\frac{1}{2}\left\{\partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} + \partial^\mu F^{\alpha\beta}\right\} = 0$$

by the first Maxwell equation, $\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0$.
Therefore

$$\partial_\nu T^{\mu\nu}_{\text{Maxwell}} = -F^{\mu\rho}J_\rho, \quad \text{and, if} \quad T^{\mu\nu} = T^{\mu\nu}_{\text{particles}} + T^{\mu\nu}_{\text{Maxwell}}, \qquad \partial_\nu T^{\mu\nu} = 0 \quad !$$

Another property carried by a particle is its angular momentum about a given point. If we ignore any contributions from intrinsic spin, we have $\vec{L} = \vec{x}_n\times\vec{p}_n$. The 3-current of such an object might then be expected to be

$$\mathcal{M}^{ijk}(x) = \sum_n\left(x^i_n p^j_n - x^j_n p^i_n\right)\delta^3\left(x-x_n\right)\frac{dx^k_n}{dt} = x^iT^{jk}(x) - x^jT^{ik}(x).$$

To make 4-dimensional we simply define

$$\mathcal{M}^{\mu\nu\rho}(x) = x^\mu T^{\nu\rho}(x) - x^\nu T^{\mu\rho}(x),$$

and

$$\partial_\rho\mathcal{M}^{\mu\nu\rho} = \delta^\mu_\rho T^{\nu\rho} + x^\mu\underbrace{\partial_\rho T^{\nu\rho}}_{0} - \delta^\nu_\rho T^{\mu\rho} - x^\nu\underbrace{\partial_\rho T^{\mu\rho}}_{0} = T^{\nu\mu} - T^{\mu\nu} = 0$$

as T is symmetric. Thus $\mathcal{M}^{\mu\nu\rho}$ corresponds to a conserved quantity, assuming T falls off sufficiently fast at ∞. Then the integral of the $\rho=0$ component is a conserved

$$L^{ij}(t) = \int d^3x\mathcal{M}^{ij0} = \text{angular momentum}.$$

We also have

$$L^{0k}(t) = \int d^3x\left(tT^{k0} - x^kT^{00}\right) = tp^k - \int x^kT^{00}d^3x.$$

Note the energy-weighted center of mass: $\bar{x}^k = \dfrac{\int x^kT^{00}d^3x}{E}$, so

$$L^{0k} = tp^k - \bar{x}^kE = E\left(\bar{x}^k - v^kt\right),$$

where $v^k = p^k/E$. Thus the conservation of L^{0k}, along with $\vec{p}$ and E, indicates that

$$\bar{x}^k(t) = \text{const} + v^k t,$$

or the center of energy moves with a velocity given by the usual formula in terms of the total momentum and energy.

$\mathcal{M}$ is not truly a tensor because it varies under translations, as does L. A translation-invariant object[5] may be formed from L, $M^2 = -p^\alpha p_\alpha$:

$$W_\alpha := MS_\alpha := \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}L^{\beta\gamma}p^\delta,$$

which is the spin. As L and p are conserved if there are no external forces, so are M and S_α. M is the total mass of the system, which we can see, is just the integrated energy density in the inertial coordinate system in which $\vec{p} = 0$. S is the spin. W is invariant under a translation $x \to x + a$,

$$L'^{\mu\nu} = \int \left((x+a)^\mu T^{\nu 0} - (x+a)^\nu T^{\mu 0}\right) = L^{\mu\nu} + a^\mu p^\nu - a^\nu p^\mu,$$

$$\text{so} \qquad MS_\alpha \to \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\left(L^{\beta\gamma}p^\delta + a^\beta p^\gamma + a^\gamma p^\beta\right)p^\delta = MS_\alpha$$

because $\epsilon_{\alpha\beta\gamma\delta}p^\gamma p^\delta = 0$.

Thus S transforms like a vector. It corresponds to the spin of the system, that is, the angular momentum in the rest frame. We would expect it to have only three components, and indeed it satisfies the constraint $p^\alpha S_\alpha = \frac{1}{2}M^{-1}\epsilon_{\alpha\beta\gamma\delta}L^{\beta\gamma}p^\alpha p^\delta = 0$.

Any isolated system has a definite value of the two scalar quantities M^2 and W^2 (and, if $M^2 \neq 0, S^2 = W^2/M^2$) which are invariants under Lorentz transformations. These play a fundamental role in classifying the possible forms of quantum fields. Because spin is quantized, $S^2 = n(n+1)\hbar^2$ after quantization, and fields must transform as some representation of the Lorentz group.

[5] W_α is known as the Pauli-Lubański vector, and its square is, along with M^2, one of the two Casimir operators of the Poincaré group, that is, operators made from the generators $L_{\mu\nu}$ and P^μ which commute with all elements of the group.