Physics 464/511 Lecture J Fall, 2016

We have considered various second order linear ordinary differential equations which emerge from separation of variables in many physics contexts. We have employed the concepts of weighted inner products and self-adjointness, and considered functions satisfying the self-adjoint differential equations

$$\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x) + \lambda w(x)u(x) = 0.$$

Here is a table of several that are important in physics and applied mathematics, in a table mostly taken from Arfken & Weber, 6th Edition, p 625.

Equation	a	b	$p(x)$	$q(x)$	λ	$w(x)$
Legendre	-1	1	$1-x^2$	0	$\ell(\ell+1)$	1
Associated Legendre	-1	1	$1-x^2$	$-\frac{m^2}{1-x^2}$	$\ell(\ell+1)$	1
Chebyshev I	-1	1	$\sqrt{1-x^2}$	0	n^2	$(1-x^2)^{-1/2}$
Chebyshev II	-1	1	$(1-x^2)^{3/2}$	0	$n(n{+}2)$	$\sqrt{1-x^2}$
Ultrasphical or Gegenbauer	-1	1	$(1-x^2)^{\alpha+1/2}$	0	$n(n{+}2\alpha)$	$(1-x^2)^{\alpha-1/2}$
Bessel	0	a	x	$-n^2/x$	a^2	x
Laguerre, $0 \leq \infty$	0	∞	xe^{-x}	0	α	e^{-x}
Associated Laguerre $k \in \mathbf{N}$	0	∞	$x^{k+1}e^{-x}$	0	$\alpha - k$	$x^k e^{-x}$
Hermite	$-\infty$	∞	e^{-x^2}	0	2α	e^{-x^2}
Harmonic Osc.	$-\pi$	π	1	0	n^2	1

Table of Self-Adjoint ODE's

We see that quite a number of them have $q = 0$, for which a constant is a solution (with $\lambda = 0$) and for which we might perhaps have polynomial solutions. Let us investigate that.

1 The "Classical" orthogonal polynomials

Suppose we have a weight function $w > 0$ on (a, b), with $\int_a^b w(x)x^n\,dx$ defined for all positive $n \in \mathbb{R}$. Then we can define a sequence of orthogonal polynomials $f_n(x)$ of order n such that

$$\int_a^b w(x) f_n(x) f_m(x)\,dx = h_n \delta_{mn}.$$

This can be done iteratively by a kind of Schmidt diagonalization.

One can show that the f's obey a recursion relation $f_{n+1} = (a_n + xb_n)f_n - c_n f_{n-1}$. The way to see this is to choose b_n such that the x^{n+1} term in $f_{n+1} - xb_n f_n$ cancels. Note $b_n \neq 0$. Then $f_{n+1} - xb_n f_n$ is a polynomial of order $\leq n$, so $= \sum_0^n \gamma_i f_i$. Then let

$$\gamma_i = \frac{1}{h_i} \int w\, f_i (f_{n+1} - xb_n f_n)\,dx.$$

For $i \leq n$, the first term vanishes directly. For $i < n - 1$, xf_i is of order $< n$, so $= \sum_{j=0}^{n-1} \delta_j f_j$, and so

$$\gamma_i = -\frac{b_n}{h_i} \sum_0^{n-1} \delta_j \int w\, f_n f_j = 0 \quad \text{for } i < n-1.$$

So the γ_i vanish except for $a_n := \gamma_n$ and $c_n := -\gamma_{n-1}$. And we have that any set of orthogonal polynomials obeys a recursion relation

$$f_{n+1} = (a_n + xb_n)f_n - c_n f_{n-1}.$$

Consider our eigenvalue problem, known as the Sturm-Liouville equation, but with $q(x) = 0$:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \lambda w(x) y(x) = 0.$$

We suppose $p(a) = p(b) = 0$, $p(x) > 0$ and $w(x) > 0$ on (a, b).

Let us also *assume* there exists a complete set of orthogonal polynomials f_n which solve this equation with eigenvalues λ_n. These are called the "classical" sets of orthogonal polynomials

Then we can show the f_n' are an orthogonal set of polynomials with weight $p(x)$.

$$\text{For} \qquad \int_a^b f_m \frac{d}{dx}\left(p(x)\frac{df_n}{dx}\right) = -\lambda_n \int f_m w f_n = -\lambda_n h_n \delta_{mn}$$

$$= \underbrace{f_m p(x)\left.\frac{df_n}{dx}\right|_a^b}_{=0} - \int_a^b f_m'(x) f_n'(x) p(x)\,dx$$

$$\text{so} \qquad \int_a^b f_m'(x) f_n'(x) p(x)\,dx = \lambda_n h_n \delta_{mn}$$

[Note: this proves $\lambda_n > 0$ for $n > 0$].

It turns out that all these classical polynomials reduce to three cases, according to whether the interval has finite or infinite endpoints.

	a	b	w	Name	$g = p/w$	Restrictions
$P_n^{(\alpha,\beta)}(x)$	-1	1	$(1-x)^\alpha(1+x)^\beta$	Jacobi	$1-x^2$	$\alpha > -1$, $\beta > -1$
$L_n^{(\alpha)}(x)$	0	∞	$x^\alpha e^{-x}$	Generalized Laguerre	x	$\alpha > -1$
$H_n(x)$	$-\infty$	∞	e^{-x^2}	Hermite	1	

Notice that $p(x)$ and $w(x)$ are always closely related.

If f_n is a solution to $\dfrac{d}{dx}g(x)w(x)\dfrac{dy}{dx} + \lambda_n w(x)y(x) = 0$, we see immediately

$$n = 0 \qquad \lambda_0 = 0$$

$$n = 1 \qquad f_1 = k_1 x + k_1', \qquad \frac{k_1}{w(x)}\frac{d}{dx}g(x)w(x) = -\lambda_1 f_1(x)$$

In fact, this form can be generalized into an expression called *Rodrigues' Formula*[1]

$$f_n(x) = \frac{1}{a_n w(x)}\frac{d^n}{dx^n}\left(w g^n\right). \tag{1}$$

It is not even obvious that f_n is a polynomial, but if you examine the cases you will see that each $\dfrac{1}{w}\dfrac{d}{dx}w$ has or makes a simple pole where g has

[1] See Supplementary Notes "Rodrigues' Formula and Orthogonal Polynomials".

a zero, and when multiplied by g is proportional to x + constant. Thus by induction f_n is a polynomial of order n.

One also makes use of $\frac{d^m}{dx^m}(wg^n) \underset{a \text{ or } b}{\longrightarrow} 0$ for $m < n$ to show that f_n is orthogonal to any polynomial of order $< n$, so in particular $\langle f_n, f_m\rangle = h_n \delta_{nm}$.

To see that f satisfies Eq. (1) note that $\frac{d}{dx} g(x) w(x) = -\lambda_1 \frac{f_1(x)}{k_1} w(x)$. Now

$$\left(\frac{d}{dx}\right)^{n+1} g \frac{d}{dx} w g^n = g \left(\frac{d}{dx}\right)^{n+2} w g^n + (n+1)\frac{dg}{dx}\left(\frac{d}{dx}\right)^{n+1} w g^n + \frac{n(n+1)}{2}\frac{d^2 g}{dx^2}\left(\frac{d}{dx}\right)^{n} w\, g^n \tag{2}$$

as g has no higher derivatives (g is quadratic).

Note $g\frac{d}{dx} w g^n = \frac{d(wg)}{dx} g^n + (n-1) w g^n \frac{dg}{dx} = w g^n \left(-\frac{\lambda_1 f_1}{k_1} + (n-1)\frac{dg}{dx}\right)$
The term in parenthesis is linear, so at most one of the $n+1$ derivatives acts on it, and

$$\left(\frac{d}{dx}\right)^{n+1} g \frac{d}{dx} w g^n = \left[-\frac{\lambda_1}{k_1} f_1 + (n-1)\frac{dg}{dx}\right]\left(\frac{d}{dx}\right)^{n+1} w g^n + (n+1)\left(-\lambda_1 + (n-1)\frac{d^2 g}{dx^2}\right)\left(\frac{d}{dx}\right)^{n} w g^n. \tag{3}$$

Equating the right hand sides of Eqs. (2) and (3) with $a_n\, w\, f_n = d^n(wg^n)/dx^n$,

$$\left\{ g\left(\frac{d}{dx}\right)^2 + \left(2\frac{dg}{dx} + \frac{\lambda_1}{k_1} f_1\right)\frac{d}{dx} + (n{+}1)\left(\frac{2{-}n}{2}\frac{d^2 g}{dx^2} + \lambda_1\right)\right\} a_n w f_n = 0,$$

or

$$\left\{ g w \frac{d^2}{dx^2} + \underbrace{\left(2g\frac{dw}{dx} + 2w\frac{dg}{dx} + \frac{\lambda_1}{k_1} w f_1\right)}_{\frac{d}{dx} g w}\frac{d}{dx} + \lambda_n w\right\} f_n = 0$$

with

$$\lambda_n = g\frac{1}{w}\frac{d^2 w}{dx^2} + \left(2\frac{dg}{dx} + \frac{\lambda_1}{k_1} f_1\right)\frac{1}{w}\frac{dw}{dx} + (n{+}1)\left[\frac{2{-}n}{2}\frac{d^2 g}{dx^2} + \lambda_1\right].$$

$$\begin{aligned} \text{From} \quad -\frac{\lambda_1}{k_1} f_1 &= \frac{1}{w}\frac{d}{dx} g w = g' + g\frac{1}{w}\frac{dw}{dx} \\ -\lambda_1 &= g'' + g\frac{1}{w}\frac{d^2 w}{dx^2} \underbrace{-g\left(\frac{1}{w}\frac{dw}{dx}\right)^2 + g'\frac{1}{w}\frac{dw}{dx}}_{2g'\frac{1}{w}\frac{dw}{dx} - \frac{\lambda_1}{k_1} f_1 \frac{\lambda_1}{k_1}} \end{aligned}$$

so $\lambda_n = -\lambda_1 - g'' + (n+1)\lambda_1 - (n+1)\left(\frac{n}{2} - 1\right) g'' = n\lambda_1 - \frac{1}{2} n(n-1) g''$ and

$$\left(\frac{d}{dx} p \frac{d}{dx} + \lambda_n w\right) f_n = 0.$$

The three classical polynomials we have considered have special cases, with different names and normalizations. Some important ones are

- Legendre: $P_n(x) = P_n^{(0,0)}(x)$ occurs in the angular equations in spherical coordinates.
- Chebyshev: $T_n(x) \propto P_n^{(-1/2,-1/2)}(x)$ is used in data fitting to minimize maximum errors.
- Gegenbauer: $C_n^{\alpha}(x) \propto P_n^{(\alpha-1/2,\alpha-1/2)}(x)$ are the spherical harmonics in four dimensions.
- Laguerre: $L_n = L_n^{(0)}$.

The associated Laguerre and the Hermite polynomials arise in the quantum mechanical bound states of the hydrogen atom and harmonic oscillator, respectively. They are also used in numerical analysis for some very efficient integration techniques.