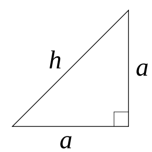
Physics 464/511 Lecture H Fall, 2015

1 Review of Complex Variables

In ancient Greece, it was believed that all real numbers (actually all ratios of lengths) were rational numbers. When the Pythagorians discovered this was not so, they regarded the new "irrational" numbers as so unnatural that only members of their priestly band could cope with the dark secret of their existence. So top-secret was the existence of these irrational numbers that when one of their members leaked this information to the general public, he was put to death. Probably the funny association of "rational" and "irrational" as properties of numbers and of human minds stems from that time.

The Pythagorians found irrationals by trying to solve the equation $x^2 = 2$. That the rationals were an incomplete set in that this equation had no solution therein clearly did not mean no solution existed, because it could be constructed geometrically as h/a.



The real numbers are similiarly incomplete with respect to the equation $x^2 = -1$. The solutions we call $\pm i$. The "existence" of i is not realized by ratios of lengths or other classical objects. But one can create an algebra for the *complex* variables, things of the form $z = x + iy$, where x and y are real numbers, which is consistent and useful. Still, i troubled people philosophically and hence we call it an "imaginary" number. But all numbers are imaginary — it is just harder to find physical objects which use the rules of complex arithmetic than objects using integer arithmetic.

The algebra assumes i commutes with everything, and $i^2 = -1$. Any polynomial or power series in z can thus be evaluated,

$$f = f(z) = \sum_i a_i z^i = f(x, y)$$

and is a complex number (or function) $\text{Re } f + i \text{ Im } f$.

The magnitude of z is defined as $|z| = \sqrt{x^2 + y^2}$, a positive (or 0) real number. The *complex conjugate* of $z = x + iy$ is $z^* = x - iy$. The argument ϕ of z is such that $x = |z| \cos\phi$, $y = |z| \sin\phi$.

From the power series expansions of e^z and for $\sin\phi$ and $\cos\phi$, we find

$$e^{i\phi} = \cos\phi + i \sin\phi.$$

Functions:

In a sense, a complex valued function of a complex variable $f(z)$ can be any pair of real valued functions u and v of two real variables

$$f(x + iy) = u(x, y) + iv(x, y).$$

The notion of (x, y) and (u, v) corresponding to complex variables is only useful if the functions satisfy certain properties. In particular, think of the derivative,

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

Assuming u and v are differentiable, this is

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x + i\Delta y}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)\frac{\Delta y}{\Delta x}}{1 + i\Delta y/\Delta x}$$

The derivative is meaningful as a complex quantity only if the limit does not depend on the direction $\Delta y/\Delta x$, which means the coefficient must vanish. This requires both the real part and the imaginary part to vanish, which gives the *Cauchy-Riemann* conditions

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

If these are satisfied we say f has the derivative

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

If, for some small region in the (x, y) plane around z_0, $f(z)$ is differentiable in this sense, we say $f(z)$ is *analytic* at $z = z_0$. If $f(z)$ is analytic at every point in a region D of the *complex plane*, we say it is analytic in D. If $f(z)$ is analytic in the whole complex plane, then $f(z)$ is an *entire* function.

Note any function $f(z)$ expressed in terms of elementary functions of z, (not of x and y separately) will satisfy the Cauchy-Riemann equations whenever the partial derivatives exist, and will therefore be analytic there.

Note that the complex conjugation function $f(z) := z^* = x - iy$ is **NOT analytic**. Neither is $z \mapsto |z| := \sqrt{x^2+y^2}$. Recall that z^* is called the complex conjugate of z and $|z|$ is called the absolute value or magnitude of z.

1.1 Contour Integrals

If $f = u + iv$,

$$\int_C f(z)dz := \int [u(x,y) + iv(x,y)]\,(dx + idy) = \int u\,dx - v\,dy + i\int u\,dy + v\,dx.$$

For the first term, think of f as a vector $\vec{A} = (u, -v, 0)$, where the third component is just added in for convenience. For the second, let $\vec{B} = (v, u, 0) = \hat{e}_3 \times \vec{A}$.

$$\begin{aligned}\int_C f(z)dz &= \int \vec{A}\cdot d\vec{r} + i\left(\int \vec{B}\cdot d\vec{r}\right)\\ &= \int_S \left(\vec{\nabla}\times\vec{A} + i\vec{\nabla}\times\vec{B}\right)_z d\sigma\\ &= \int_S \left\{-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right\} d\sigma = 0\end{aligned}$$

which vanishes due to the Cauchy-Riemann conditions, if f is analytic on the surface S bounded by the closed curve C.

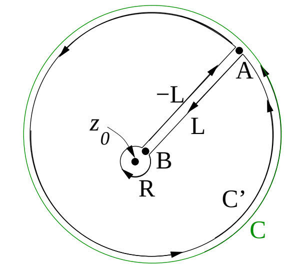
This theorem:

$\oint_C f(z)dz = 0$ if f is analytic within the area of the complex plane S bounded by the contour C

is called the *Cauchy Integral Theorem*.

Now consider $\oint_C \frac{f(z)}{z-z_0}dz$, where f is analytic on S, but z_0 is a point within S, so of course there is a pole from the $\frac{1}{z-z_0}$.

First consider the contour C' which consists of C, followed by L, R, and $-L$, where $-L$ means we retrace the same path in the opposite direction. R is a circle of small radius ρ centered on z_0. Now C' is a closed path enclosing the area S' which does not include the point z_0, and so has no singularities of $f(z)/(z-z_0)$, so



$$0 = \oint_{C'} \frac{f(z)}{z-z_0}dz = \oint_C + \int_L + \oint_R + \int_{-L}.$$

Now $\int_L = \int_A^B (\ )dz$ while $\int_{-L} = \int_B^A (\ )dz = \int_A^B (\ )(-dz) = -\int_A^B (\ )dz$, so

$$\oint_C \frac{f(z)}{z-z_0}dz = -\oint_R \frac{f(z)}{z-z_0}dz.$$

On R, $z - z_0 = \rho e^{-i\phi}$ and $dz = -i\rho e^{-i\phi}d\phi$ so

$$\begin{aligned}\oint_C \frac{f(z)}{z-z_0}dz &= -\int_0^{2\pi} \frac{f(z_0+\rho e^{-i\phi})}{\rho e^{-i\phi}}(-i\rho)e^{-i\phi}d\phi\\ = i\int_0^{2\pi} f(z_0+\rho e^{-i\phi})\,d\phi\end{aligned}$$

Now we can choose ρ arbitrarily small, and as f is assumed analytic (and therefore continuous) at $z = z_0$, as $\rho \to 0$, $f \to f(z_0)$, a constant, and $\int d\phi = 2\pi$, so

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

for f analytic within C and z_0 within C. This is called the *Cauchy integral formula*. It is a very powerful tool, for it tells us that the value of f within some region on which it is analytic is determined by the value of f on the boundary:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z}.$$

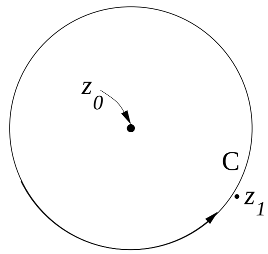
We can also evaluate the derivatives:

$$f^{(n)}(z) = \frac{1}{2\pi i}\oint_C f(z')\,dz'\left(\frac{d}{dz}\right)^n \frac{1}{z'-z} = \frac{n!}{2\pi i}\oint_C f(z')\,dz'\,(z'-z)^{-n-1}.$$

This expression is perfectly well defined for all n. We will call it the *Cauchy differentiation formula*. We see that an analytic function is infinitely differentiable on the open set D on which it is analytic.

A consequence of the Cauchy integral formula is that a power series for an analytic function about z_0 converges within the largest circle around z_0 which contains only analytic points of f.

Let z_1 be the closest nonanalytic point, and C a circle of radius just less than $|z_1 - z_0|$. Then f is analytic within and on C, so for z within C,



$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z}.$$

But $z' - z = (z' - z_0)\left(1 - \dfrac{z - z_0}{z' - z_0}\right)$, and as $\left|\dfrac{z - z_0}{z' - z_0}\right| < 1$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)\left(1 - \dfrac{z-z_0}{z'-z_0}\right)} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_C \frac{f(z')\,dz'\,(z-z_0)^n}{(z'-z_0)^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \end{aligned}$$

In other words, we've proven that $f(z)$ is given by a power series expansion in z within any circle around z_0 of radius less than $|z_1 - z_0|$.

1.2 Analytic Continuation

Now we will show an amazing manifestation of Blake's statement that every grain of sand reflects the whole universe. Well — at least for analytic functions.

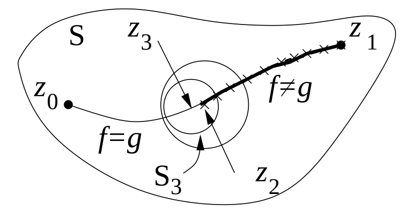
Let f be a function analytic on a connected open set S, and L be a short line segment (of non-zero length) contained in S. Then f is completely determined by its values on L.

That is, if f and g are analytic in S and $f = g$ on L, $f = g$ everywhere in S.

The proof uses the concept of analytic continuation as well as concepts from topology. Start with a circle C within S about a point z_0 on L. Then f and g have power series expansions which converge within C to the same values, so $f - g$ has a power series expansion which converges to zero on L. Thus each term in the expansion must be zero, and $f = g$ within C.

Now I claim $f = g$ everywhere in S. If not, there exists a z_1 such that $f(z_1) \neq g(z_1)$ with z_1 in S. Connect z_0 to z_1 by a path lying within S.

We have seen that for part of this path, $f = g$. The interval along this path where $f = g$ must end somewhere, say at $z = z_2$. Draw a circle around z_2 which lies within S, and then pick a point z_3 on the $f = g$ part of the path close to z_2, and draw a circle S_3 about it, including z_2 and within the first circle. Then again the power series for $f - g \to 0$ about z_3, so its coefficients are all zero and it converges within S_3 to zero, and $f = g$ within S_3, But that includes the point z_2, so $f(z_2) - g(z_2)$, in contradiction with the assumption.



Therefore $f = g$ all along the path, and hence everywhere within S. Note that f, given originally within a small circle, can be extended from each circle to each overlapping circle, so the bounds of the region where a function is analytic can be extended until some singularity makes the power series' circle of convergence not grow.

Example: The Γ function was defined for real $z > 0$ by $\Gamma(z) = \displaystyle\int_0^\infty e^{-u}u^{z-1}du$. Using the complex exponential

$$u^{z-1} = e^{(z-1)\ln u} = e^{(x-1)\ln u}e^{iy\ln u},$$

we see that the integral is absolutely convergent if $x > 0$. And the derivative with respect to z is also defined in that region. But away from the real axis we may continue to negative $\operatorname{Re} z$.

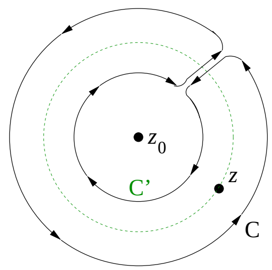
Thus $\Gamma(z)$ is an analtyic function in the half plane $\operatorname{Re} z > 0$ satisfying the analytic relationship $\Gamma(z) = \dfrac{1}{z}\Gamma(z+1)$. We may analytically continue this relation to *evaluate* $\Gamma(z)$ for $x > -1$ except for $z = 0$. Then we may use it again and again, until we have determined $\Gamma(z)$ everywhere except at the negative integers and zero. Our extension is analytic in each region, so by our theorem this is the unique analytic continuation of the Γ function.

Suppose $f(z)$ is analytic in some connected region R which includes an open interval of the x axis, and suppose $f(z)$ is real on that interval. Then $f(z) = f^*(z^*)$ everywhere that z and z^* are in R.

Proof: The Cauchy-Riemann equations are preserved under $v \to -v$, $y \to -y$, so $g(z) := f^*(z^*)$ is an analytic function which coincides with $f(z)$ on a line segment (the interval of the x axis). Thus $f = g$ everywhere one can get by analytic continuation, which includes all of R.

1.3 Laurent Series

Sometimes we have a function analytic in an annulus, say $R_1 < |z-z_0| < R_2$. Then by writing a Cauchy integral for $f(z)$ with the solid contour as shown, as $f(z) = \oint_{C-C'} \frac{dz'}{2\pi i}\frac{f(z')}{z'-z}$ we may expand $\frac{1}{z'-z} = \frac{1}{z'-z_0}\left[1 - \frac{z-z_0}{z'-z_0}\right]^{-1} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}}$ on C where $|z-z_0| < |z'-z_0|$, and



$$\frac{1}{z'-z} = -\frac{1}{z-z_0}\left[1 - \frac{z'-z_0}{z-z_0}\right]^{-1} = -\sum_{n=1}^{\infty}\frac{(z'-z_0)^{n-1}}{(z-z_0)^n} \quad \text{on} \quad C',$$

where $|z'-z_0| < |z-z_0|$.

With $b_n = \oint_C \frac{dz'}{2\pi i}\frac{f(z')}{(z'-z_0)^{n+1}}$ and $c_n = \oint_{C'} \frac{dz'}{2\pi i} f(z')(z'-z_0)^{n-1}$ we have

$$f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n + \sum_{m=1}^{\infty}\frac{c_m}{(z-z_0)^m}.$$

We can combine these by writing

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad \text{with} \quad a_n = \begin{cases} b_n & \text{for } n \geq 0 \\ c_{-n} & \text{for } n < 0 \end{cases}$$

which is called a *Laurent series*. The coefficients are given by

$$a_n = \frac{1}{2\pi i}\oint \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

with the integral around a full circle in the annulus.

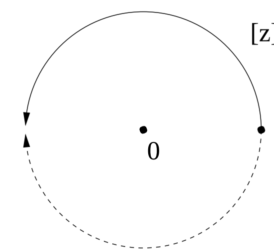
If the function is analytic in the region inside the annulus as well, all the a_n for $n \leq -1$ are zero, for $f(z')(z'-z^0)^{|n|-1}$ is now analytic and the $\oint$ of it is zero. If f has a *simple pole* at $z = z_0$, a_{-1} is nonzero but all other a_n for negative n are zero. $f(z) = g(z)/(z-z_0)$ for $g(z)$ analytic is an example.

If $f(z) = \frac{g(z)}{(z-z_0)^N}$ for g analytic and $g(z_0) \neq 0$, then $a_n = 0$ for $n < -N$, but $a_{-N} \neq 0$. Other negative $n's$ may or may not have zero a_n's. We say $f(z)$ has a *pole of order* N at z_0.

From the Laurent expansion alone, we cannot conclude what is happening if nonzero a_{-n}'s continue to arbitrarily large n. For example, the Laurent series for $\frac{1}{z-\frac{1}{2}}$ found from the annulus $|z| = 1$ has such an infinite number of terms. But if $f(z)$ is analytic in some neighborhood of z_0 except at z_0, and the Laurent expansion comes from within this neighborhood, then if a_n is nonzero for infinitely many $n < 0$, $f(z)$ has an *essential singularity* at z_0, which is an obscenity which we shall try to avoid. (the famous e^{-1/x^2} at $x = 0$ is one such.)

1.4 Branch points

Consider $f(z) = z^{1/2}$. How should that be defined? For real positive z we know we want a real positive number. If $z = re^{i\theta}$, then $f(z) = \sqrt{r}\,e^{i\theta/2}$ meets the requirements. But now consider starting from $z = 1$, and continuously varying around the circle $z = e^{i\theta}$ until you reach $z = -1$, $\theta = \pi$, $f = e^{i\pi/2} = i$. Good enough, $i = \sqrt{-1}$ is what we started complex variables with. But what if we go continuously along the dashed path $z = e^{-i\phi}$, $\phi : 0 \to \pi$, so $f \to e^{-i\pi/2} = -i$. Two different values of f at the same point!



To avoid this, we place an arbitrary *cut* in the complex plane, and say that $f(z)$ has a *discontinuity* along that line.

Note that placing the cut along the negative axis is only a convention. For the square root function the cut can be along any curve from zero to infinity. But the ends cannot be moved: they are called *branch points* and are singular points, not because f blows up there, but because it is not analytic there. [*e.g.* $\frac{\partial f}{\partial z} \sim z^{-1/2}$ blows up.]

If we describe the square-root function $u = \sqrt{z}$ with the cut along the negative real axis, as above, in terms of the magnitude and argument ϕ_z of z, we see that $|u| = \sqrt{|z|}$ and $\phi_u = \frac{1}{2}\phi_z$ with $-\pi < \phi_z \leq \pi$, so $-\pi/2 < \phi_u \leq \pi/2$. But z is equally well defined with $0 \leq \phi_z < 2\pi$, and then we would have $\tilde{u} = \sqrt{|z|}$ continuous along the negative axis, but with $\tilde{u}(x-iy) = -u(x-iy)$ for positive x and y, as $\tilde{\phi}_u = \phi_u + \pi$ in that region. We see that we might extend the square-root function's domain to $0 \leq |u|$ and $-2\pi \leq \phi_u \leq 2\pi$ constrained[1] to be periodic in ϕ_u with period 4π. Of course each complex number z appears twice in the extended domain. If we make a copy of the z plane, and cut the two copies and glue them together so as to make f a continuous function, analytic everywhere except at $z = 0$, we get this space. Of course the gluing can't take place without the sheets passing through each other transparently, so our extended complex plane is a manifold not quite embeddable in $\mathbb{R}^3$. This space is called a *Riemann surface.*

Just like the square root, all other fractional powers of z have a branch point at zero and a cut which is usually taken along the negative real axis. For $z^{p/q}$, with $p, q \in \mathbb{N}^+$, we need a q-sheeted *Riemann surface.*

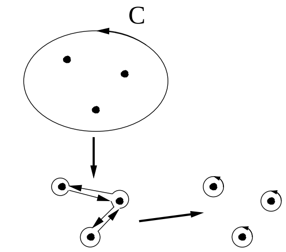
The function $(z+1)^{1/2}$ has its branch point at $z = -1$. We take the cut along $x \leq -1, y = 0$, but this is only by convention.

Now consider $f(z) = (z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$. If I lay both cuts to the left, we have a line along which $f(z)$ is discontinuous. But if we examine the discontinuity for $z = x \pm i0$, $x < -1$, we find there is no discontinuity there, because each factor has contributed a phase of i above the cut (total -1) while each gives $-i$ (total -1) below, for continuity. So in fact we have a branch cut from -1 to $+1$.

Suppose $f(z)$ is analytic within some contour except at z_0. Then $\oint f(z)dz = 2\pi i a_{-1}$, where a_n is the coefficient of the Laurent expansion of f about z_0.

[1] I am skipping fine points about $|u| = 0$.

a_{-1} is called the *residue* of f at z_0. If f has several isolated singularities, the contour C can be shrunk until it decomposes into separate contours around each singularity, and

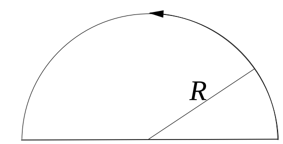


$$\oint_C f(z)\,dz = 2\pi i \sum_{z_i} \operatorname{Res} f$$

where the sum is over the singularities at z_i within C.

Examples:

$I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$. Consider the contour consisting of the real line from $-R$ to R, and a half circle $\mathcal{R}$ as shown, closing the contour. On $\mathcal{R}$, $z = Re^{i\phi}$, $dz = Re^{i\phi} i\,dz$. Then $\displaystyle\oint \frac{1}{1+z^2} dz = I + \lim_{R\to\infty} \int_0^\pi \frac{R\,e^{i\phi}\,id\phi}{1+R^2e^{2i\phi}}$. As $R \to \infty$, the integrand $\sim R^{-1} \to 0$, $\int_{\mathrm{R}}$ vanishes, so



$$I = 2\pi i \sum_{\text{UHP}} \operatorname{Res} \frac{1}{1+z^2}$$

The only singularities of $\dfrac{1}{1+z^2} = \dfrac{1}{2i}\left(\dfrac{1}{z-i} - \dfrac{1}{z+i}\right)$ are at $\pm i$, only $+i$ being in the upper half plane, within the contour, and the residue of $\frac{1}{z-i}$ is 1, so

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 2\pi i \frac{1}{2i} = \pi.$$

This formula indicates an interesting fact. Consider a contour integral of f around C. If I move the contour continuously, it makes no difference as long as I don't sweep over a pole, but it changes discontinuously if I do. Now consider what happens if there is a pole on the contour, as in

$$I = \int_{-a}^{a} \frac{f(x)}{x}\, dx,$$

where f is analytic for $|z| \leq 1$. The integral is not well defined in a normal Riemann sense, because of the behavior at $x = 0$. Close the contour with a semicircle C: $z = ae^{i\theta}$, $0 \le \theta \le \pi$, so $\int_C + I = \oint \frac{f(z)}{z}\,dz$. It is, of course, still not defined, although the $\int_C$ part is fine.

Now consider displacing the pole slightly,

$$\oint \frac{f(z)}{z - i\epsilon}\,dz.$$

Now the contour includes the pole and the integral is $2\pi i f(i\epsilon)$, and as $\epsilon \to 0$, I includes a piece $2\pi i f(0)$.

Now consider displacing the pole slightly in the other direction, $\int \frac{f(z)}{z + i\epsilon} dz$. Now the contour does not include a singularity near 0, and does not include the $2\pi i f(0)$ piece.

The $i\epsilon$ prescriptions are each well defined, although they give different answers. We can also take the average, called the *principal part*

$$P\int_{-a}^{a} \frac{f(x)\,dx}{x} := \frac{1}{2}\left[\int_{-a}^{a} \frac{f(x)\,dx}{x + i\epsilon} + \int_{-a}^{a} \frac{f(x)\,dx}{x - i\epsilon}\right].$$

It can also be shown (Arfken 2nd ed page 352) that this is the value you would get if you took

$$\lim_{\delta \to 0^+}\left[\int_{-a}^{-\delta} \frac{f(x)}{x}\,dx + \int_{\delta}^{a} \frac{f(x)}{x}\,dx\right],$$

excluding **symmetrically** a region about the pole.

For our next example I need the concepts of fourier transforms. The fourier transform of a function $f(x), -\infty \le x \le \infty$ is the function $\tilde f(k) := \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-ikx}$. We will see in Lecture M that this transform can be inverted $f(x) := \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde f(k) e^{+ikx}$, under appropriate conditions that these integrals are well defined. For this to be true, we must have

$$\int_{-\infty}^{\infty} e^{ik(x-x')}dk = 2\pi\delta(x - x').$$

We will justify this rather ill-defined statement in Lecture M.

Suppose we want a solution of the 1-D wave equation

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = s(x,t),$$

where I have included a source term. If we make a Fourier transform[2] in both variables,

$$g(k,\omega) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{i\omega t} f(x,t),$$

then $(-\omega^2 + c^2k^2)\, g(k,\omega) = \tilde s(k,\omega) := \frac{1}{2\pi}\int dx\,dt\,e^{-ikx+i\omega t} s(x,t)$. The solutions are clearly given by $g(k,\omega) = \tilde s/(c^2k^2 - \omega^2)$.

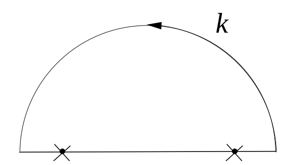
Suppose the source is a point source at $x = 0$ oscillating with a fixed frequency $s(x,t) = e^{-i\omega_0 t}\delta(x)$, so

$$\tilde s = \int \frac{dx\,dt}{2\pi}\, e^{-ikx}\delta(x)\, e^{i(\omega-\omega_0)t} = \int \frac{dt}{2\pi}\, e^{i(\omega-\omega_0)t} = \delta(\omega - \omega_0).$$

Then

$$\begin{aligned} f(x,t) &= \int \frac{dk\,d\omega}{2\pi}\, e^{ikx-i\omega t} g(k,\omega) \\ &= \int \frac{dk\,d\omega}{2\pi}\, e^{ikx-i\omega t}\, \frac{\delta(\omega-\omega_0)}{c^2k^2 - \omega^2} \\ &= \int \frac{dk}{2\pi}\, e^{ikx-i\omega_0 t} \frac{1}{c^2k^2 - \omega_0^2}. \end{aligned}$$

Note that the integral is not well defined because of the poles right on the axis of integration.



$$\frac{1}{c^2k^2 - \omega_0^2} = \frac{1}{2\omega_0 c}\left(\frac{1}{k - \omega_0/c} - \frac{1}{k + \omega_0/c}\right).$$

For $x > 0$, we can throw in the semicircle in the U. H. P., as $\left|e^{ikx}\right| \sim e^{-x \text{ Im } k} \to 0$, so

$$f(x,t) = \frac{ie^{-i\omega_0 t}}{2\omega_0 c} \text{Res}\left(e^{ikx}\left[\frac{1}{k - \omega_0/c} - \frac{1}{k + \omega_0/c}\right]\right),$$

[2]Physicists insist on changing the sign of the exponent $ikx \to -i\omega t$ when the variable is time rather than space. Of course mathematicians variables are unphysical and can't tell the difference.

but for each pole we must decide whether to use a $\pm i\epsilon$ or a principal part, or some other prescription for handling the pole. The $1/(k - \frac{\omega_0}{c} - i\epsilon)$ gives

$$f(x,t) = \frac{i}{2\omega_0 c} e^{-i\omega_0(t-x/c)},$$

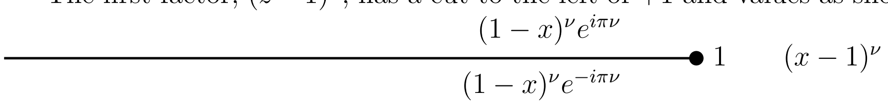
a right-going wave, while if we use $1/(k-\omega_0/c+i\epsilon)$ this gives no contribution. The $1/(k+\omega_0/c)$ gives the opposite wave.

The ambiguity in whether to include the poles on the axis has a physical origin — we are trying to determine the wave consistent with certain sources, but any solution to the source-free or homogeneous equation could be added and we would still have a solution to the equation with the specified source. The poles at $k = \pm\omega_0/c$ produce such solutions.

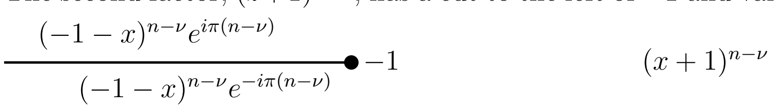
1.5 The Beta Function

Now we turn to a very different example. We saw that $\sqrt{z^2-1}$ has a branch cut which has branch points at ± 1. We could choose to cut it from -1 to $+1$. This is also true of the function $f(z) = (z-1)^\nu (z+1)^{n-\nu}$. We discuss the cuts due to each factor, and then combine them.

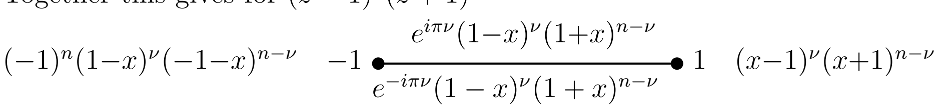
The first factor, $(z-1)^\nu$, has a cut to the left of $+1$ and values as shown:



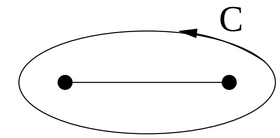
The second factor, $(z+1)^{n-\nu}$, has a cut to the left of -1 and values as shown:



Together this gives for $(z-1)^\nu (z+1)^{n-\nu}$



with no cuts outside the interval $[-1,+1]$.



If we consider integrating this function around a contour C, by collapsing we get

$$\begin{aligned}
\oint_C f(z)\,dz &= \int_{-1}^{1} (1-x)^\nu (1+x)^{n-\nu} \times \left\{ e^{i\pi\nu}(-dx) + e^{-i\pi\nu}(dx) \right\} \\
&= -2i \sin(\pi\nu) \int_{-1}^{1} (1-x)^\nu (1+x)^{n-\nu} dx.
\end{aligned}$$

The *Euler Beta function* is usually defined by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt \qquad \text{for Re } x > 0, \text{ Re } y > 0$$

but with $u = 2t-1$, this becomes

$$B(x,y) = \int_{-1}^{1} du (1+u)^{x-1}(1-u)^{y-1} 2^{1-x-y}.$$

Thus

$$B(n+1-\nu, 1+\nu) = \frac{2^{-n-2}\, i}{\sin(\pi\nu)} \oint (z-1)^\nu (z+1)^{n-\nu}\, dz.$$

Now deform the contour until it is a very large circle

$$\begin{aligned}
\oint (z-1)^\nu (z+1)^{n-\nu}\, dz &= \oint z^n\, dz \left(1 - \frac{1}{z}\right)^\nu \left(1 + \frac{1}{z}\right)^{n-\nu} \\
&= \oint_U u^{-n-2} du (1-u)^\nu (1+u)^{n-\nu} \\
&= \frac{2\pi i}{(n+1)!} \left(\frac{d}{du}\right)^{n+1} (1-u)^\nu (1+u)^{n-\nu} \Bigg|_{u=0},
\end{aligned}$$

where I have substituted $u = 1/z$ so the contour U is a small circle about 0, and used the Cauchy Differentiation Formula. The simplest case, $n = -1$, gives

$$B(-\nu, 1+\nu) = -\frac{\pi}{\sin(\pi\nu)}.$$

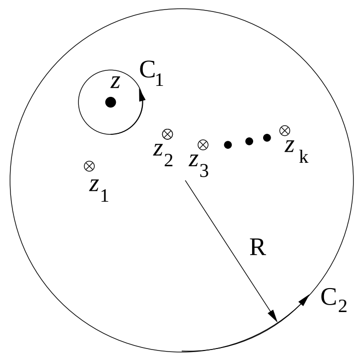
We shall see, in Lecture K

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

$$\text{so} \qquad B(-\nu, 1+\nu) = \frac{\Gamma(-\nu)\Gamma(1+\nu)}{\Gamma(1)} = -\frac{\pi}{\sin(\pi\nu)},$$

$$\text{or} \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

1.6 Mittag Leffler



Suppose $f(z)$ is analytic everywhere except at a set of isolated simple poles at z_k and $f(z)/z^n \underset{|z|\to\infty}{\longrightarrow} 0$. Let $r_k = \operatorname*{Res}_{z=z_k} f(z) = \lim_{z\to z_k}(z-z_k)f(z)$. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z'-z}dz' \\ &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z'-z}dz' - \sum_k \operatorname*{Res}_{z_k} \frac{f(z')}{z'-z} \end{aligned}$$

As $f(z')$ has only a simple pole at z_k, and $\frac{1}{z'-z}$ is analytic there (for $z \neq z_k$),

$$\operatorname*{Res}_{z_k} \frac{f(z')}{z'-z} = \frac{1}{z_k - z} r_k.$$

Suppose $f(z) \underset{|z|\to\infty}{\longrightarrow} 0$, so $\oint_{C_2} \frac{f(z')}{z'-z}dz' \to 0$ as $R \to \infty$. Then

$$f(z) = \sum_k \frac{r_k}{z - z_k}.$$

If $f(z) \not\to 0$ but $f(z)/z \to 0$, we cannot ignore the contour at infinity. But

$$f(z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z'-z}dz' + \sum_k \frac{r_k}{z-z_k}.$$

At zero,

$$f(0) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z'}dz' + \sum_k \frac{r_k}{-z_k},$$

$$\text{so} \quad f(z) - f(0) = \underbrace{\frac{1}{2\pi i}\oint_{C_2} f(z') \underbrace{\left(\frac{1}{z'-z} - \frac{1}{z'}\right)}_{\dfrac{z}{z'(z'-z)}} dz'}_{\to 0 \text{ as } R\ f(z)/z^2|_R \to 0} + \sum_k r_k \left(\frac{1}{z-z_k} + \frac{1}{z_k}\right).$$

This is the first term of the *Mittag-Leffler* expansion.

If $f(z)/z$ does not vanish at infinity, but f is known to have behavior at infinity bounded by a polynomial, a subtraction similar to the above, but with more terms, will work. Then $f(z)/z^n$ does vanish for some higher n, and subtracting the first n terms in a power series expansion of $f(z)$ will produce an expression in terms of the residues of the poles alone, with the more complicated integral vanishing in the limit.

1.7 Entire function with simple zeros

Now consider $g(z)$ an entire function with simple zeros at z_k. (That is, $g(z_k) = 0$ but $g'(z_k) \neq 0$.) Let $f(z) = \dfrac{1}{g}\dfrac{dg}{dz} = \dfrac{d}{dz}\ln g$. f has simple poles at z_k, with residue 1 (if $g \sim (z-z_k)h(z)$, $dg/dz|_{z=z_k} = h(z_k)$, so $\operatorname{Res} \dfrac{1}{g}\dfrac{dg}{dz} = \operatorname{Res}\dfrac{1}{z-z_k} = 1$). Assuming that f doesn't blow up as $z \to \infty$,

$$f(z) = \frac{g'(0)}{g(0)} + \sum_k \left(\frac{1}{z-z_k} + \frac{1}{z_k}\right) = \frac{d}{dz}\ln g$$

$$\int_0^z \frac{d}{dz}\ln g = \ln g(z) - \ln g(0) = \frac{g'(0)}{g(0)}z + \sum_k \ln\frac{z-z_k}{-z_k} + \frac{z}{z_k},$$

and

$$g = g(0)e^{g'(0)z/g(0)} \prod_k \left[e^{z/z_k}\left(1 - \frac{z}{z_k}\right)\right],$$

so we have a product expansion of g.

Example:

$$\begin{aligned} g &= \frac{\sin(z)}{z}, \qquad g(0) = 1, \quad g' = 0, \quad \frac{\sin z}{z} = \prod_{n\neq 0} e^{z/n\pi}\left(1 - \frac{z}{n\pi}\right) \\ &= \prod_{n=1}^{\infty} e^{z/n\pi}e^{z/(-n\pi)}\left(1-\frac{z}{n\pi}\right)\left(1-\frac{z}{-n\pi}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right). \end{aligned}$$

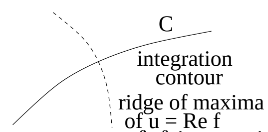
so

$$\sin z = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right) \qquad \text{as promised.}$$

I am going to skip the inversion of a power series expansion and the stability of amplifiers, although they are nice things. But of much more use is a method of deriving asymptotic expansions called the

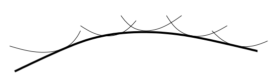
1.8 Method of Steepest Descents

Let us consider $I(s) = \int_C g(z)e^{sf(z)}dz$ for real s. For large positive s, the function will be dominated by any region at which Re f takes on its maximum value along the contour. Assume f and g are analytic where necessary, and let $f = u + iv$. Then the integrand would appear to have a contribution $\sim e^{s \max_C u}$. But the contour C can be deformed so that the maximum of u takes on a different value, without changing the integral. What is happening is that, if the imaginary part of f is varying, the phase e^{isv} is varying rapidly, and we are not getting a contribution as large as we think because of these contributions.



Suppose we choose the contour to go over the ridge $u =$ maximum at the lowest point, the saddle point. Then

$$\left.\frac{\partial u}{\partial x}\right|_{\rm sp} = \left.\frac{\partial u}{\partial y}\right|_{\rm sp} = 0, \qquad \text{so} \qquad \left.\frac{df}{dz}\right|_{\rm sp} = 0.$$



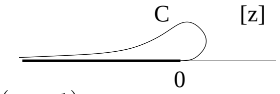
This gives the lowest estimate $e^{s \max u}$ but this is a good estimate, because $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ also, so the phase is not changing, and there are no cancelations.

As the region around the maximum dominates,

$$\begin{aligned} I(s) \;&\approx\; \int_C g(z_0)e^{s\left[f(z_0) + \frac{1}{2}(z-z_0)^2 f''(z_0)\right]}\,dz \\ &= g(z_0)e^{sf(z_0)} \underbrace{\int e^{\frac{s}{2}(z-z_0)^2 f''(z_0)}\,dz}_{\sqrt{\frac{2\pi}{-sf''(z_0)}}}. \end{aligned}$$

Example: The Hankel function $H_\nu^{(1)}(s)$ for positive s is defined as

$$H_\nu^{(1)}(s) = \frac{1}{i\pi}\int_C e^{(s/2)(z-z^{-1})}\frac{dz}{z^{\nu+1}}.$$



Near 0, we go out along the real axis, so $e^{\frac{s}{2}\left(z-z^{-1}\right)} \approx e^{-s/2z} \to 0$ faster than $1/z^{\nu+1}$ blows up. As $z \to -\infty$ it is also well defined.

The maximum of $f = \frac{1}{2}\left(z - z^{-1}\right)$ is at $\frac{1}{2} + \frac{1}{2}z^{-2} = 0$, or $z = i$. So we let C go thorugh i, and get $f = i$, $f'' = -z^{-3} = -i$, so

$$H_\nu^1(s) \sim \frac{i^{-\nu-1}}{\pi i}\,e^{is}\sqrt{\frac{-2\pi}{s(-i)}} = \pm\sqrt{\frac{2}{\pi}}\,\frac{e^{is}}{\sqrt{s}}\,i^{-\nu-5/2}.$$

The overall sign requires closer inspection, as given in Arfken, showing the correct sign is $-$.

Finally, we will get the leading term in Stirling's approximation to $s!$ for large positive s. $s! = \int_0^\infty \nu^s e^{-\nu}d\nu$. Let $\nu = sz$, so

$$s! = s^{s+1}\int_0^\infty z^s e^{-sz}dz = s^{s+1}\int_0^\infty e^{s(\ln z - z)}dz.$$

The saddle point $f' = 0$ is when $\frac{d}{dz}(\ln z - z) = \frac{1}{z} - 1 = 0$, so $z_0 = 1$, $\quad f''(z_0) = -1/z^2 + 0 = -1$, $\quad f(z_0) = -1$, so

$$\int_0^\infty e^{s(\ln z - z)}dz \approx e^{-s}\sqrt{\frac{-2\pi}{s(-1)}} = \sqrt{\frac{2\pi}{s}}\,e^{-s}$$

where the positive sign is clear from taking the integral of a positive integrand along the real axis. Thus

$$s! \sim \sqrt{2\pi s}\,s^s\,e^{-s}.$$

This is Stirling's formula, but we will derive an asymptotic series later, of which this is the first term. The extra terms, however, are hardly ever needed.