Physics 464/511 Lecture G Fall, 2016

1 Infinite Series

We will now review some mathematical techniques which are generally useful in deriving expressions or approximations to solutions of ordinary differential equations. Infinite series expansions and complex variables are the basic tools. We will then turn to special functions. Towards the end of the term we will return to more general techniques.

An infinite series, written $\displaystyle\sum_{n=1}^{\infty} u_n$, is a formal limit of a sequence of partial sums $s_i = \displaystyle\sum_{n=1}^{i} u_n$. The sequence s_i converges to S ($\lim_{i\to\infty} s_i = S$) if

$$\forall\epsilon > 0, \exists N \ni \forall n > N, |s_i - S| < \epsilon.$$

A series can converge only if $u_n \to 0$, but that is not enough. A series is *absolutely convergent* if $\sum |u_n|$ is convergent. $\sum 1/n$ diverges, but $\sum 1/n^{1.01}$ converges. There are many tests, increasingly subtle, to find whether or not a series converges. I will not discuss them unless I need them — you have seen some of them in your math classes, I assume.

Sometimes the terms in our series will be functions rather than numbers. Then $\sum u_n(\vec{r}) = S(\vec{r})$ in some domain means that, for each $\vec{r}$ in that domain, the sequence of numbers converges.

Sequences of functions can have peculiar properties. For example, the sequence of functions $\quad u_n(x) = \begin{cases} 1 & \text{for } n < x < n+1, \\ & = 0 \text{ otherwise.} \end{cases}$, converges at every point to zero,

$$\lim_{n\to\infty} u_n(x) = 0, \quad \text{but} \quad \lim_{n\to\infty}\int_{-\infty}^{\infty} u_n(x) = 1.$$

To enable us to use reasonable mathematics, one defines the more restrictive notion of *uniform convergence*:

$$s_n \to S \text{ uniformly in } D \text{ if } \forall\epsilon > 0, \exists N \ni \forall n > N, \forall\vec{r}\in D, |s_n(\vec{r}) - S(\vec{r})| < \epsilon.$$

Note that this differs form just plain convergence in D,

$$\forall\epsilon > 0, \forall\vec{r}\in D, \exists N \ni \forall n > N, |s_n(\vec{r}) - S(\vec{r})| < \epsilon,$$

because for uniform convergence the N has to be chosen the same for all points in the domain. Our function above converges on the whole real axis, but not uniformly.

Uniformly convergent series can be treated, term by term, for integration and differentiation or discussions of continuity.

The most important series is Taylor's

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \left(\frac{d}{dx}\right)^n f(x)\bigg|_{x=a}.$$

Any infinitely differentiable function can therefore be associated with a Taylor series about any point. Almost always that series will converge to $f(x)$ for small enough $x - a$, although there are exceptions.

The expansion of Taylor series in 3-D is

$$f(\vec{r}_0 + \vec{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\vec{a}\cdot\vec{\nabla}\right]^n f(\vec{r})\bigg|_{\vec{r}=\vec{r}_0}.$$

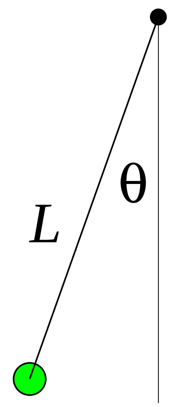
It is not wise to write this as

$$f(\vec{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[(\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\right]^n f(\vec{r})\bigg|_{\vec{r}=\vec{r}_0}$$

because the $\vec{\nabla}$'s act only on f, not on the factors $\vec{r} - \vec{r}_0$, while if we wrote $\left[(\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\right]^2$ we would mean $(\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\left[(\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\right]$, which includes a piece where the first $\vec{\nabla}$ acts on the second $\vec{r}$.

We will now see how power series can provide ways of getting an answer for a problem unsolvable in closed form.

1.1 Elliptic Integrals



Let us consider a simple pendulum, for which energy conservation states

$$\frac{1}{2}mv^2 - mgL\cos\theta = -mgL\cos\theta_M,$$

where θ_M is the maximum angle reached, and $v = L\dot\theta$. Thus

$$\dot\theta^2 = \frac{2g}{L}\left(\cos\theta - \cos\theta_M\right), \quad \text{or} \quad \frac{dt}{d\theta} = \sqrt{\frac{L}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_M}}.$$

The period is twice the time to go from $-\theta_M$ to θ_M, or four times the time to go from 0 to θ_M:

$$T = 4\int_0^{\theta_M}\frac{dt}{d\theta}d\theta = 4\sqrt{\frac{L}{2g}}\int_0^{\theta_M}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_M}}.$$

Make the substitution $\theta \to \phi$ with $\sin\theta/2 = \sin\dfrac{\theta_M}{2}\ \sin\phi$,

$$\begin{aligned} \cos\theta &= 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta_M}{2}\sin^2\phi, \\ \cos\theta_M &= 1 - 2\sin^2\frac{\theta_M}{2}, \\ \text{so}\quad \sqrt{\cos\theta - \cos\theta_M} &= \sqrt{2}\sin(\theta_M/2)\cos\phi. \end{aligned}$$

$$\begin{aligned} \text{Also}\quad -\sin\theta d\theta &= -\left(2\sin^2(\theta_M/2)\right)\left(2\cos\phi\sin\phi\, d\phi\right) \\ &= -2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta \\ &= -2\sin\frac{\theta_M}{2}\sin\phi\sqrt{1 - \sin^2\frac{\theta_M}{2}\ \sin^2\phi}\ d\theta \end{aligned}$$

so

$$d\theta = \frac{2\cos\phi\, d\phi}{\sqrt{1 - \sin^2\frac{\theta_M}{2}\ \sin^2\phi}}\sin\frac{\theta_M}{2},$$

and

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \sin^2\frac{\theta_M}{2}\ \sin^2\phi}}.$$

Define

$$K(m) = \int_0^{\pi/2}\left(1 - m\sin^2\phi\right)^{-1/2}d\phi,$$

called the *complete elliptic integral of the first kind*. Then the period of the pendulum is

$$T = 4\sqrt{\frac{L}{g}}\,K\left(\sin^2\frac{\theta_M}{2}\right).$$

Giving this integral a name doesn't necessarily help us know what it is. Let us get a power series expansion for K for small m. Using the binomial theorem

$$\left(1 - m\sin^2\phi\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-m)^n\sin^{2n}\phi.$$

The binomial coefficient

$$\binom{-1/2}{n} = \frac{-\frac{1}{2}\times -\frac{3}{2}\times\cdots\left(-n+\frac{1}{2}\right)}{n!} = \frac{(2n-1)!!(-1)^n}{2^n\, n!},$$

where $(2n-1)!!$ means the product of all the odd integers from 1 to $2n-1$, (and $0! = (-1)!! = 1$).

We also need

$$\int_0^{\pi/2}\sin^{2n}\phi\, d\phi = \frac{1}{4}\int_0^{2\pi}\sin^{2n}\phi\, d\phi.$$

Let us use $\sin\phi = (2i)^{-1}\left(e^{i\phi} - e^{-i\phi}\right)$. Then

$$\begin{aligned} \frac{1}{4}\int_0^{2\pi}\sin^{2n}\phi\, d\phi &= 2^{-2n-2}(-1)^n\int_0^{2\pi}d\phi\left(e^{i\phi} - e^{-i\phi}\right)^{2n} \\ &= 2^{-2n-2}\sum_r\int_0^{2\pi}\binom{2n}{r}\left(e^{i\phi}\right)^{2n-r}(-1)^{n-r}\left(e^{-i\phi}\right)^r d\phi \\ &= 2^{-2n-2}\sum_r(-1)^{r-n}\binom{2n}{r}\int_0^{2\pi}e^{i\phi(2n-2r)}d\phi \end{aligned}$$

But $$\int_0^{2\pi} e^{im\phi} d\phi = \frac{1}{im}\, e^{im\phi}\Big|_0^{2\pi} = 0 \quad \text{for } m \in \mathbb{Z}, m \neq 0,$$
and is 2π for $m = 0$, so the only term in the sum which survives is $r = n$, and
$$\int_0^{\pi/2} \sin^{2n}\phi d\phi = \binom{2n}{n} 2^{-2n-1}\pi = \frac{(2n)!}{n!\,2^n}\frac{\pi}{n!\,2^{n+1}} = \frac{(2n-1)!!\,\pi}{n!\,2^{n+1}}.$$
Finally
$$K = \int_0^{\pi/2} \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n m^n \sin^{2n}\phi\, d\phi = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 m^n.$$

We have derived a power series expansion for $K(m)$. The radius of convergence can be found by the ratio test, for
$$\frac{a_{n+1}}{a_n} = \left[\frac{(2n+1)!!}{(2n-1)!!\,(n+1)\,2}\right]^2 m = \frac{(2n+1)m}{2n+2} \to m$$
and hence the series converges for $m < 1$ and diverges for $m > 1$.

We should have expected this, because $\left(1 - m\sin^2\phi\right)^{-1/2}$ develops a singularity for $m > 1$ in the range $\sin^2\phi \leq 1$ which is integrated over.

What happened to our feeling learned in elementary courses that the pendulum was simple? This was based on the small angle approximation $\theta \approx \sin\theta$, or $\sin(\theta_M/2)$ small, m small. As $m \to 0$, $K \to \pi/2$ and $T \to 2\pi\sqrt{L/g}$ as it says in elementary textbooks.

The elliptic functions, of which K is one, occur in many different contexts in physics. Moving the square root in $K(m)$ from the denominator to the numerator gives the *complete elliptic integral of the second kind*, discussed in your homework, $E(m) = \int_0^{\pi/2}\sqrt{1 - m\sin^2\theta}\, d\theta$. The integral
$$u(\phi) := \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$
when inverted to give $\phi(u)$ gives rise to functions known as
$$\begin{aligned} sn(u) &= \sin(\phi) \\ cn(u) &= \cos(\phi) \\ dn(u) &= \sqrt{1 - m\sin^2\phi} \end{aligned}$$

and many other combinations These are called the Jacobi elliptic functions, and are characterized by being analytic functions of a complex variable u, which are periodic under $u \to u + 4K$ and also under $u \to u + 4iK'$. They have a simple pole in each unit cell.

Related to these are the Jacobi Theta functions ϑ_i, for $i = 1, 2, 3, 4$, which arise in solving electrostatics problems by the method of images. I first discovered them in work on a model of elementary particles in which each particle is a piece of string!

1.2 Generating functions, Bernoulli

One use of a power series is a concept called a generating function $G(t, z)$, which can be expanded in a power series in t. Thus
$$G(t, z) = \sum_{n=0}^{\infty} P_n(z) t^n$$
for an infinite set of functions $P_n(z)$. An example we shall investigate later is $G = (1 - 2tz + t^2)^{-1/2}$, where $P_n(z)$ is the "n'th Legendre polynomial". Right now we will start with a generating function for a set of numbers, rather than functions,
$$G(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$
B_n are called the *Bernoulli numbers*. They arise in power series expansions of trigonometric functions, in integration techniques, and are related to the Riemann zeta function.

From $\displaystyle\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ we see
$$t = (e^t - 1)\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \sum_{m=1}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$
Equating terms of the same power of t, $B_0 = 1$ and from t^{p+1},
$$\sum_{n=0}^{p} \frac{1}{n!} B_n \frac{1}{(p-n+1)!} = 0.$$

This can be used recursively

$$B_p = -(p!) \sum_{m=0}^{p-1} \frac{B_m}{m!(p-m+1)!},$$

so $B_1 = -(1!)\dfrac{B_0}{0!\,2!} = -\dfrac{1}{2}$, $B_2 = -2!\left(\dfrac{B_0}{0!\,3!} + \dfrac{B_1}{1!\,2!}\right) = \dfrac{1}{6}$, *etc.*.

If we set $t = 2ix$, $\dfrac{2ix}{e^{2ix}-1} = \displaystyle\sum_{m=0}^{\infty} B_m \frac{(2x)^m}{m!} i^m$. Subtract $2ixB_1$:

$$\begin{aligned} \frac{2ix}{e^{2ix}-1} + ix &= \sum_{\substack{m\neq 1\\ m=0}}^{\infty} B_m \frac{(2x)^m}{m!} i^m \\ &= ix\left(\frac{e^{2ix}+1}{e^{2ix}-1}\right) = ix\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = x\cot x. \end{aligned}$$

But $x\cot x$ is an even function of x, so all the B_m for odd m on the right must be zero. Thus

$$B_{2n+1} = 0 \quad \text{except } B_1 = -1/2$$

$$x\cot x = \sum_{n=0}^{\infty} B_{2n} \frac{(2x)^{2n}(-1)^n}{(2n)!}.$$

The Bernoulli numbers are connected to the Riemann zeta function

$$\zeta(x) := \sum_{p=1}^{\infty} p^{-x}, \qquad x > 1.$$

The connection is

$$B_{2n} = -\frac{(-1)^n\, 2\,(2n)!}{(2\pi)^{2n}}\,\zeta(2n). \tag{1}$$

We can generalize the Bernoulli numbers into Bernoulli functions, by using the generating function

$$\frac{te^{ts}}{e^t-1} = \sum_{n=0}^{\infty} B_n(s)\frac{t^n}{n!}.$$

Clearly when $s = 0$ we have just the Bernoulli numbers $B_n = B_n(0)$. In expanding the exponential in powers of t on the left hand side, the stuff from e^{ts} has one power of s for each power of t, but powers of t from expanding the denominator have no powers of s. Therefore

$$B_n(s) \text{ is a polynomial of degree } n.$$

Differentiating with respect to s,

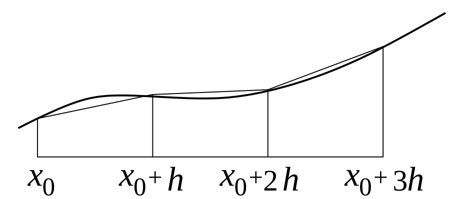
$$\begin{aligned} \frac{t^2e^{ts}}{e^t-1} &= \sum_{n=0}^{\infty} B_n'(s)\frac{t^n}{n!} \\ &= \sum_{r=0}^{\infty} B_r(s)\frac{t^{r+1}}{r!} \Longrightarrow B_n'(s) = nB_{n-1}(s). \end{aligned}$$

Now let $s = 1-u$.

$$\sum_n B_n(u)\frac{t^n}{n!} = \frac{te^{t-ts}}{e^t-1} = \frac{(-t)e^{(-t)s}}{e^{(-t)}-1} = \sum_n B_n(s)\frac{(-t)^n}{n!}$$

$$\text{or} \qquad B_n(1-s) = (-1)^n B_n(s).$$

A major application of Bernoulli polynomials is in developing the Euler-Maclaurin integration formula. When integrations cannot be done in closed form, one often seeks expansions of some type or another. For numerical integration. one always uses a finite number of values of the argument of the function. If these values are evenly spaced, one method is to use the trapezoid rule



$$\int f(x)\,dx \sim h\left[\frac{1}{2}f(x_0) + f(x_0+h) + f(x_0+2h) + \frac{1}{2}f(x_0+3h)\right],$$

which approximates the area by trapezoids. Corrections for each trapezoid $\int_a^b f(x)\,dx$ from $\frac{1}{2}(f(a)+f(b))(b-a)$ can be found as follows. Consider the interval $[0,1]$

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_0^1 f(x)B_0(x)\,dx = \int_0^1 f(x)\frac{d}{dx}B_1(x)\,dx \\ &= f(x)B_1(x)|_0^1 - \int B_1(x)\frac{d}{dx}f(x) = \frac{1}{2}f(0) + \frac{1}{2}f(1) - \int B_1 f'(x)\,dx. \end{aligned}$$

The error is now expressed as an integral of $f' = f^{(1)}$

$$\int_0^1 f(x)\,dx = \frac{1}{2}f(0) + \frac{1}{2}f(1) \quad \underbrace{-\int \frac{1}{2}\left(\frac{d}{dx}B_2\right) f'(x)\,dx}$$
$$-\frac{1}{2}\,B_2(x)f^{(1)}(x)\Big|_0^1 + \frac{1}{2}\int B_2(x)f^{(2)}(x)dx$$
$$= -\frac{1}{2}B_2\left[f^{(1)}(1) - f^{(1)}(0)\right] + \frac{1}{2}\int B_2(x)f^{(2)}(x)dx$$

One can continue to replace B_n by $\frac{1}{n+1}B'_{n+1}$ and integrate by parts, developing a series of correction terms involving higher derivatives of f at the endpoints only, and an error term involving a high order derivative of f,

$$\int_0^1 f(x)\,dx = \frac{1}{2}[f(0)+f(1)] - \sum_{p=1}^{q}\frac{B_{2p}}{(2p)!}\left[f^{(2p-1)}(1) - f^{(2p-1)}(0)\right] + \frac{1}{(2q)!}\int f^{(2q)}(x)B_{2q}(x)\,dx.$$

What is the optimal q to stop at will depend on the smoothness of f.

This formula does not depend in any way on the interval being from 0 to 1 except for the error term. If we wish to integrate from 0 to n by using the formula for each unit interval, we get

$$\int_0^n f(x)\,dx = \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n) - \sum_{p=1}^{q}\frac{B_{2p}}{(2p)!}\left[f^{(2p-1)}(n) - f^{(2p-1)}(0)\right] + \text{remainder}.$$

Note this is the extended trapezoid rule, that the correction terms from the endpoints are still no more complicated. Note also the dropped remainder term vanishes if f is a polynomial of order less than $2q$.

While we are considering the zeta function, let us say a few quick words about cute properties of it.

Recall that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Now each n has a unique decomposition into prime numbers $p_1^{n_1}p_2^{n_2}\ldots$. A sum over all integers is the same as a sum over all values of n_1, n_2, ... from 0 to ∞, so

$$\zeta(s) = \sum_{n_1,n_2,\ldots}\prod_{i=1}^{\infty}\left(p_i^{-s}\right)^{n_i}.$$

The factors are independent,

$$\zeta(s) = \prod_{i=1}^{\infty}\sum_{n_i=0}^{\infty}\left[\left(p_i^{-s}\right)^{n_i}\right] = \prod_{i=1}^{\infty}\frac{1}{1-p_i^{-s}}$$

This sort of clever rearrangement resulting in a sum over peculiar classes of integers (here only primes) is a typical tool in number theory.

As a place where Bernoulli numbers or zeta functions enter Physics, consider the historically first law of quantum mechanics, Planck's law for the energy density per unit frequency range per unit volume of photons in an empty box is

$$u(\omega)\,d\omega = \frac{\hbar}{\pi^2c^3}\,\frac{\omega^3\,d\omega}{e^{\hbar\omega/kT}-1}.$$

Substituting $x = \hbar\omega/kT$, we see that the total energy/unit volume is not infinite, as Wien would have told us, but

$$\frac{(kT)^4}{\pi^2\hbar^3c^3}\int_0^\infty\frac{x^3\,dx}{e^x-1}.$$

You might be tempted to associate this directly with the Bernoulli expansion, *e.g.* as $\int_0^\infty \sum B_n \frac{x^{n+2}}{n!}dx$, but each term therein is divergent. Instead, rewrite as

$$\int_0^\infty dx\,x^3e^{-x}\left(1-e^{-x}\right)^{-1} = \sum_{n=1}^{\infty}\int_0^\infty dx\,x^3e^{-nx} = \sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^\infty ds\,s^3e^{-s}.$$

We shall see in half a lecture that the integral is $\Gamma(4) = 3! = 6$, so

$$\int_0^\infty\frac{x^3\,dx}{e^x-1} = 6\sum_{n=1}^{\infty}\frac{1}{n^4} = 6\zeta(4) = -6\times\frac{(2\pi)^4}{2\cdot 4!}B_4.$$

$B_4 = -1/30$, so the answer is $\pi^4/15$, and the energy density is

$$\frac{\pi^2}{15\hbar^3c^3}(kT)^4.$$

Do Homework 5 problem 3 by a similar method. Hint: $\binom{-2}{n} = (-1)^n(n+1)$.

1.3 Infinite products

Occasionally one finds a useful representation of a function not as an infinite series $f = \sum a_n$ but as an infinite product

$$f = \prod_{n=1}^{\infty} f_i.$$

Just as for series, f is defined as the limit of the partial products. Clearly convergence requires $f_i \to 1$. In fact, writing

$$\ln f = \sum_{n=1}^{\infty} \ln f_i$$

reduces an infinite product to an infinite series, requiring $\ln f_i \to 0$ sufficiently fast, so $f_i \to 1$ sufficiently fast.

Consider

$$x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

For any x, $|\ln f_n| \to \frac{x^2}{\pi^2}\frac{1}{n^2}$, so the product converges. Clearly the product is zero whenever $x = \pm n\pi$ for any integer n, including zero. Reminiscent of $\sin x$? That's exactly what it is. We will prove later

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

If you accept this reasonable formula for the time being, we can prove some others. Note we can derive a formula for $\cos x = \sin 2x/2\sin x$ by

$$\begin{aligned}
\sin 2x &= 2x \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{n^2\pi^2}\right) \\
2\sin x &= 2x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2\pi^2}\right) = 2x \prod_{m=1}^{\infty} \left(1 - \frac{4x^2}{(2m)^2\pi^2}\right) \\
&= 2x \prod_{\text{even } n>0}^{\infty} \left(1 - \frac{4x^2}{n^2\pi^2}\right)
\end{aligned}$$

$$\prod_{\text{all } n=1}^{\infty} \Big/ \prod_{\text{even } n>0}^{\infty} = \prod_{\text{odd } n>0}^{\infty}, \text{ so}$$

$$\cos x = \prod_{\text{odd } n>0}^{\infty} \left(1 - \frac{4x^2}{n^2\pi^2}\right) = \prod_{m=1}^{\infty} \left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right).$$

A more important formula for us is one for $x\cot x$ (page 7), which we know in terms of the Bernoulli numbers. Taking the ln of our product expansion for the sine,

$$\ln\sin(x) = \ln(x) + \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Differentiating with respect to x gives

$$\frac{\cos x}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{-2x}{n^2\pi^2} \frac{1}{\left(1 - \dfrac{x^2}{n^2\pi^2}\right)},$$

$$\begin{aligned}
\text{or} \quad x\cot x &= 1 - 2\sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2} \sum_{r=0}^{\infty} \left(\frac{x^2}{n^2\pi^2}\right)^r \\
&= 1 - 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{x}{n\pi}\right)^{2m}.
\end{aligned}$$

Now we do the sum on n for each m,

$$x\cot x = 1 - 2\sum_{m=1}^{\infty} \left(\frac{x}{\pi}\right)^{2m} \zeta(2m).$$

But $x\cot x = \sum_{m=0}^{\infty} (-1)^m B_{2m} \frac{(2x)^{2m}}{(2m)!}$. Equating like powers of x, $B_0 = 1$ and

$$B_{2m} = (-1)^{m+1} 2\frac{(2m)!}{(2\pi)^{2m}} \zeta(2m), \qquad m \geq 1$$

as we claimed earlier (Eq. 1, page 7).

1.4 The gamma and incomplete gamma functions, and asymptotic expansions

We define the *incomplete gamma function* as

$$\Gamma(a,x) = \int_x^\infty e^{-u} u^{a-1} du$$

which is defined for $x > 0$ for all a, because an exponential blows up faster than any power. For $a > 0$, the limit $x \to 0$ exists. We also define

$$\Gamma(a) = \Gamma(a,0) = \int_0^\infty e^{-u} u^{a-1} du$$

which is called the *gamma function*. Note $\Gamma(1) = \int_0^\infty e^{-u} du = 1$, and, for $a > 0$,

$$\begin{aligned} a\Gamma(a) &= \int_0^\infty e^{-u} a u^{a-1} du = \int_0^\infty e^{-u} \left(\frac{d}{du} u^a\right) du \\ &= e^{-u} u^a \Big|_0^\infty - \int_0^\infty u^a \left(\frac{d}{du} e^{-u}\right) du \\ &= 0 + \int_0^\infty u^a e^{-u} du = \Gamma(a+1). \end{aligned}$$

Thus by induction, $\Gamma(n+1) = n\Gamma(n) = n(n-1)\cdots 1 \cdot \Gamma(1) = n!$.

Thus the Γ function is a generalization of the factorial, but defined for any real argument greater than zero, rather than only for integers.

Note I have fulfilled my promise to show $\int_0^\infty ds\, s^3 e^{-s} = \Gamma(4) = 6$.

Note that the integral $\Gamma(z) = \int_0^\infty e^{-u}\, u^{z-1}\, du$ is well defined for $\operatorname{Re} z > 0$, even if z is complex, and the proof we have just given that $z\Gamma(z) = \Gamma(z+1)$ is also valid for $\operatorname{Re} z > 0$. We may extend the definition of $\Gamma(z)$ to all complex z other than 0 or negative integers by using the integral for $\Gamma(z+n)$ with $n > \operatorname{Re} z$, and then recursing to negative values with $\Gamma(u) = \frac{1}{u}\Gamma(u+1)$. This is a form of analytic continuation, which we will discuss in the next lecture.

Now let us return to the real topic of discussion, deriving an asymptotic

series for the incomplete Γ:

$$\begin{aligned} \Gamma(a,x) &= \int_x^\infty e^{-u} u^{a-1} du \\ &= -\int_x^\infty \left(\frac{d}{du} e^{-u}\right) u^{a-1} du \\ &= -\, u^{a-1} e^{-u} \Big|_x^\infty + (a-1) \int_x^\infty u^{a-2} e^{-u} du \\ &= x^{a-1} e^{-x} + (a-1)\Gamma(a-1,x). \end{aligned}$$

Repeating this process n times gives

$$\begin{aligned} \Gamma(a,x) &= e^{-x}\Big(x^{a-1} + (a-1)x^{a-2} + (a-1)(a-2)x^{a-3} + \ldots \\ &\qquad + (a-1)(a-2)\cdots(a-n+1)x^{a-n}\Big) \\ &\quad + (a-1)(a-2)\cdots(a-n)\Gamma(a-n,x). \end{aligned}$$

If we carry out the process indefinitely and discard the remainder, we seem to get an infinite series. The series does not converge, however, because the ratio test of the $n+1$'st term $(-1)^n \dfrac{(n-a)!x^{a-n-1}}{(-a)!}$ to the n'th $(-1)^{n-1}\dfrac{(n-a-1)!x^{a-n}}{(-a)!}$ is $-\dfrac{n-a}{x} \underset{n\to\infty}{\longrightarrow} -\infty$ for all x.

Thus the infinite sum is not useful, but for finite n, the remainder has a definite sign, as $\Gamma(a-n,x) > 0$ for real a, and this sign oscillates for $n > a$. So successive partial sums bound the correct answer. In fact, the error in dropping the remainder after n terms is

$$\begin{aligned} \left|\frac{(n-a)!}{(-a)!}\Gamma(a-n,x)\right| &< \left|\frac{(n-a)!}{(-a)!}\right| \int_x^\infty u^{a-n-1} du \\ &= \left|\frac{(n-a-1)!}{(-a)!}\right| x^{a-n}. \end{aligned}$$

For a fixed n and large x, this remainder can still be small. For any x, there is an ideal $n \approx a + x$, at which to stop.

When $a = 0$

$$E_1(x) := \Gamma(0,x) = \int_x^\infty \frac{e^{-u}}{u}\, du$$

is a version of the *logarithmic integral function*. We have

$$E_1(x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots\right).$$

If we have a series $f(x) = \sum_{i=0}^{n} a_i x^{-i} + R_n(x)$, with $R_n(x)x^{p+n} \xrightarrow[x\to\infty]{} 0$ for some fixed p, we say $f \sim \sum a_i x^{-i}$ is an asymptotic expansion.