Physics 464/511 Homework #7
Due: Nov. 7, 2016 at 5:00 P. M.

1 [10 pts] Solve the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

by direct series substitution.

(a) Verify that the indicial equation is $k(k-1) = 0$.

(b) Using $k = 0$, obtain a series of even powers of x $(a_1 = 0)$.

$$y_{\text{even}} = a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots \right],$$

where $a_{j+2} = \dfrac{j(j+1) - n(n+1)}{(j+1)(j+2)} a_j$.

(c) Using $k = 1$, develop a series of odd powers of x, $(a_1 = 0)$

$$y_{\text{odd}} = a_0 \left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots \right],$$

where $a_{j+2} = \dfrac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)} a_j$.

(d) Show that both solutions, y_{even} and y_{odd}, diverge for $x = \pm 1$ *if the series continues to infinity.*

(e) Finally, show that by an appropriate choice of n, one series at a time may be converted into a polynomial, thereby avoiding the divergence catastrophe. In quantum mechanics, this restriction of n to integral values corresponds to *quantization* of angular momentum.

2 [5 pts] Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

has a regular solution $P_n(x)$ and an irregular solution $Q_n(x)$. Show that the Wronskian of $P_n(x)$ and $Q_n(x)$ is given by

$$P_n(x)Q_n'(x) - P_n'(x)Q_n(x) = \frac{A_n}{1-x^2},$$

with A_n *independent* of x.

3 [5 pts] $U_n(x)$, the Chebyshev polynomial (type II) satisfies the differential equation

$$(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0.$$

(a) Locate the singular points that appear in the *finite* plane and show whether they are regular or irregular.

(b) Put this equation in self-adjoint form.

(c) Identify the complete eigenvalue.

(d) Identify the weighting function.