Physics 464/511 Homework #5
Due: Oct. 24, 2016 at 5:00 P. M.

1 The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ may be represented parametrically by $x = a \sin\theta$, $y = b\cos\theta$. Show that the length of arc within the first quadrant is

$$a\, E(m) \quad \text{where} \quad E(m) := \int_0^{\pi/2} \sqrt{1 - m \sin^2\theta}\, d\theta,$$

with $0 \le m := \dfrac{a^2 - b^2}{a^2} \le 1$.

2 Derive the expansion

$$\begin{aligned} E(m) &= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{m^2}{3} - \dots\right\} \\ &= \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{m^n}{2n-1}\right\}. \end{aligned}$$

3 Prove that

$$\int_0^{\infty} \frac{x^n\, e^x\, dx}{(e^x - 1)^2} = n!\, \zeta(n)$$

for real $n > 1$, but that both sides diverge as $n \searrow 1$.