**INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025**

**Answers to Questions 4. Weds, Nov 12th**

1. (a) We begin by noting that the matrix elements for the tight-binding Hamiltonian are

$$\langle i|H|j\rangle = \begin{cases} \epsilon & (i=j) \\ -t & \text{(i, j, nearest neighbors)} \\ 0 & \text{(otherwise)} \end{cases} \tag{1}$$

where $\epsilon$ is the energy of an isolated orbital and $-t$ is the hopping matrix element. Now the orbital at site $A$, position $\mathbf{r}_j$ is connected to orbitals at sites $B$ which are located in the unit cells at positions $\mathbf{r}_j$, $\mathbf{r}_j - \mathbf{b}$ and $\mathbf{r}_j - \mathbf{a}$. (Note the minus signs). Consequently, the tight-binding Hamiltonian takes the form

$$H = \overbrace{-t\sum_j \left\{ \left[ {\psi^\dagger}_B(\mathbf{r}_i) + {\psi^\dagger}_B(\mathbf{r}_i - \mathbf{a}) + {\psi^\dagger}_B(\mathbf{r}_i - \mathbf{b}) \right] \psi_A(\mathbf{r}_i) + \text{H.c} \right\}}^{\text{hopping between nearest neighbors}} + (\epsilon - \mu) \sum_i (n_A(i) + n_B(i)). \tag{2}$$

Here ${\psi^\dagger}_{A,B}(\mathbf{r}_j)$ creates an electron at site A or B respectively in the unit cell located at position $\mathbf{r}_j$.

(b) If we Fourier transform, writing

$${\psi^\dagger}_\lambda(\mathbf{r}_j) = \frac{1}{\sqrt{N_s}} \sum_{\mathbf{k}} {c^\dagger}_{\mathbf{k}\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \qquad (\lambda = A, B) \tag{3}$$

then substituting into the Hamiltonian, we obtain

$$\begin{aligned} H &= -\frac{t}{N_s} \sum_{j,\mathbf{k},\mathbf{k}'} \left[ {c^\dagger}_{\mathbf{k}B}(1 + e^{i\mathbf{k}\cdot\mathbf{a}} + e^{i\mathbf{k}\cdot\mathbf{b}}) c_{\mathbf{k}A} \overbrace{e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_j}}^{\to N_s \delta_{\mathbf{k},\mathbf{k}'}} + \text{H.c} \right] + (\epsilon - \mu) \sum_{\mathbf{k}\lambda} {c^\dagger}_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} \\ &= \sum_{\mathbf{k}} \left( {c^\dagger}_{\mathbf{k}B}, {c^\dagger}_{\mathbf{k}A} \right) \begin{bmatrix} \epsilon - \mu & \Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & \epsilon - \mu \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}B} \\ c_{\mathbf{k}A} \end{pmatrix} \end{aligned} \tag{4}$$

where

$$\Delta(\mathbf{k}) = -t(1 + e^{i\mathbf{k}\cdot\mathbf{a}} + e^{i\mathbf{k}\cdot\mathbf{b}})$$

The eigenvalues of the matrix

$$\begin{bmatrix} \epsilon - \mu & \Delta^*(\mathbf{k}) \\ (\mathbf{k}) & \epsilon - \mu \end{bmatrix}$$

are

$$\begin{aligned} \epsilon_{\mathbf{k}\pm} &= \pm|\Delta(\mathbf{k})| + (\epsilon - \mu) \\ &= \pm t\sqrt{(3 + 2\cos(\mathbf{k}.\mathbf{a}) + 2\cos(\mathbf{k}.\mathbf{b}) + 2\cos(\mathbf{k}.(\mathbf{a} - \mathbf{b})))} + (\epsilon - \mu) \end{aligned} \tag{5}$$

(see Fig. 1)

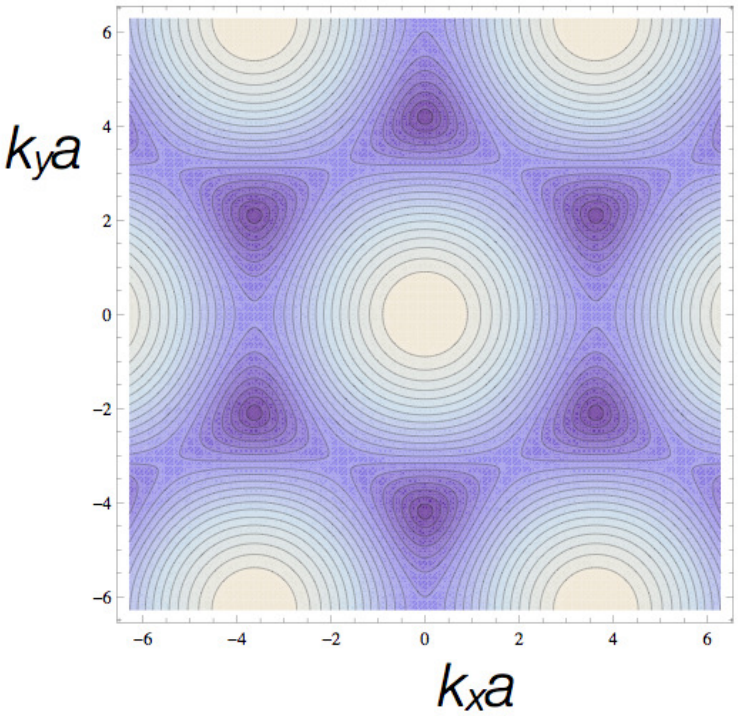


Figure 1: Contour plot of dispersion $\epsilon_{\mathbf{k}+}$ given by equation (5).

(c) If $e^{i\mathbf{K}\cdot\mathbf{a}} = e^{i\frac{2\pi}{3}}$ , $e^{i\mathbf{K}\cdot\mathbf{b}} = e^{-i\frac{2\pi}{3}}$ , then clearly,

$$\Delta(\mathbf{K}) \propto (1 + e^{i\mathbf{K}\cdot\mathbf{a}} + e^{i\mathbf{K}\cdot\mathbf{b}}) = (1 + e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}}) = 0$$

To satisfy the conditions

$$\begin{aligned} \mathbf{K}.\mathbf{a} &= \frac{\sqrt{3}}{2}(K_x a) + \frac{1}{2}K_y a = \frac{2\pi}{3} \\ \mathbf{K}.\mathbf{b} &= \frac{\sqrt{3}}{2}(K_x a) - \frac{1}{2}K_y a = -\frac{2\pi}{3} \end{aligned} \tag{6}$$

we require $K_x = 0$, $K_y = \frac{4\pi}{3a}$, or $\mathbf{K} = \frac{4\pi}{3a}\mathbf{j}$. When $\mathbf{k} = \pm\mathbf{K}$, $\Delta(\mathbf{k}) = \Delta(\pm\mathbf{K}) = 0$.

(d) If we expand $\Delta(\mathbf{p} + \mathbf{K})$ for small $\mathbf{p}$, we obtain

$$\begin{aligned} \Delta(\mathbf{k}) &= -t\big[1 + e^{i\frac{2\pi}{3}}(1 + i\mathbf{p}\cdot\mathbf{a}) + e^{-i\frac{2\pi}{3}}(1 + i\mathbf{p}\cdot\mathbf{b})\big] \\ &= -it\big[e^{i\frac{2\pi}{3}}\mathbf{p}\cdot\mathbf{a} + e^{-i\frac{2\pi}{3}}\mathbf{p}\cdot\mathbf{b}\big] \\ &= -it\big[(-\frac{1}{2} + i\frac{\sqrt{3}}{2})\mathbf{p}\cdot\mathbf{a} + (-\frac{1}{2} - i\frac{\sqrt{3}}{2})\mathbf{p}\cdot\mathbf{b}\big] \end{aligned}$$

$$= \quad it\left[\frac{\sqrt{3}}{2}p_x a - i\frac{\sqrt{3}}{2}p_y a\right] = \overbrace{\left(\frac{\sqrt{3}ta}{2}\right)}^{\tilde{c}}[p_y + ip_x] \tag{7}$$

so that we can write

$$\begin{bmatrix} \epsilon-\mu & \Delta(\mathbf{p}+\mathbf{K}) \\ \Delta^*(\mathbf{p}+\mathbf{K}) & \epsilon-\mu \end{bmatrix} \approx \begin{bmatrix} \epsilon-\mu & \tilde{c}(p_y+ip_x) \\ \tilde{c}(p_y-ip_x) & \epsilon-\mu \end{bmatrix} = (\epsilon-\mu)\underline{1} + \tilde{c}\,(p_y\sigma_x - p_x\sigma_y)$$
$$= \quad (\epsilon-\mu)\underline{1} + \tilde{c}\,(\vec{\sigma}\times\mathbf{p}) \tag{8}$$

In the vicinity of $\mathbf{k}\sim\mathbf{K}$, we thus have

$$\sum_{\mathbf{p}\sim 0}\left(c^\dagger{}_{\mathbf{p}+\mathbf{K}B}, c^\dagger{}_{\mathbf{p}+\mathbf{K}A}\right)\begin{bmatrix} \epsilon-\mu & \Delta(\mathbf{p}+\mathbf{K}) \\ \Delta^*(\mathbf{p}+\mathbf{K}) & \epsilon-\mu \end{bmatrix}\begin{pmatrix} c_{\mathbf{p}+\mathbf{K}B} \\ c_{\mathbf{p}+\mathbf{K}A}\end{pmatrix}$$
$$= \quad \sum_{\mathbf{p}\sim 0}\psi^\dagger{}_{\mathbf{p}+}\left((\epsilon-\mu)\underline{1} + \tilde{c}\vec{\sigma}\times\mathbf{p}\right)\psi_{\mathbf{p}+} \tag{9}$$

where

$$\psi_{\mathbf{p}+} = \begin{pmatrix} c_{\mathbf{p}+\mathbf{K}B} \\ c_{\mathbf{p}+\mathbf{K}A}\end{pmatrix}, \tag{10}$$

Similarly, for $\mathbf{k}=\mathbf{p}-\mathbf{K}$, where $\mathbf{p}$ is small, we have

$$\begin{bmatrix} \epsilon-\mu & \Delta(\mathbf{p}-\mathbf{K}) \\ \Delta^*(\mathbf{p}-\mathbf{K}) & \epsilon-\mu \end{bmatrix} \approx \begin{bmatrix} \epsilon-\mu & -\tilde{c}(p_y-ip_x) \\ -\tilde{c}(p_y+ip_x) & \epsilon-\mu \end{bmatrix} = (\epsilon-\mu)\underline{1} + \tilde{c}(\mathbf{p}\times\vec{\sigma}^T) \tag{11}$$

so that

$$\sum_{\mathbf{p}\sim 0}\left(c^\dagger{}_{\mathbf{p}-\mathbf{K}B}, c^\dagger{}_{\mathbf{p}-\mathbf{K}A}\right)\begin{bmatrix} \epsilon-\mu & \Delta(\mathbf{p}-\mathbf{K}) \\ \Delta^*(\mathbf{p}-\mathbf{K}) & \epsilon-\mu \end{bmatrix}\begin{pmatrix} c_{\mathbf{p}+\mathbf{K}B} \\ c_{\mathbf{p}-\mathbf{K}A}\end{pmatrix} = \sum_{\mathbf{p}\sim 0}\psi^\dagger{}_{\mathbf{p}-}\left((\epsilon-\mu)\underline{1} + \tilde{c}(\vec{\sigma}\times\mathbf{p})\right)\psi_{\mathbf{p}-} \tag{12}$$

where

$$\psi_{\mathbf{p}-} = \begin{pmatrix} c_{\mathbf{p}-\mathbf{K}A} \\ -c_{\mathbf{p}-\mathbf{K}B}\end{pmatrix}, \tag{13}$$

Combining the two contributions (9) and (12), the low energy Hamiltonian thus has the form

$$H = \sum_{\mathbf{p}\lambda}\psi^\dagger{}_{\mathbf{p}\lambda}(\tilde{c}\,(\vec{\sigma}\times\mathbf{p}) + (\epsilon-\mu)\underline{1})\psi_{\mathbf{p}\lambda} \tag{14}$$

2. (a) If we take the expression for the Free energy

$$F(\lambda) = -T\ln Z(\lambda) = -T\ln \mathrm{Tr}[e^{-\beta[H_o+\lambda V]}] \tag{15}$$

and differentiate it, we obtain

$$\frac{\partial F}{\partial\lambda} = -\frac{T}{Z}\frac{\partial Z}{\partial\lambda}. \tag{16}$$

Now

$$\frac{\partial Z}{\partial \lambda} = \mathrm{Tr}[\frac{\partial e^{-\beta[H_o+\lambda V]}}{\partial \lambda}] = -\beta \mathrm{Tr}[V e^{-\beta[H_o+\lambda V]}]] \tag{17}$$

so that

$$\frac{\partial F}{\partial \lambda} = \frac{\mathrm{Tr}[V e^{-\beta[H_o+\lambda V]}]]}{Z} = \langle V \rangle = \langle V_{int} \rangle / \lambda. \tag{18}$$

(b) By integrating the result of part (a) over $\lambda$, we obtain

$$\Delta F = \int_0^1 d\lambda \frac{\partial F}{\partial \lambda} = \int_0^1 \frac{d\lambda}{\lambda} \langle V_{int}(\lambda) \rangle \tag{19}$$

(c) If the interaction energy has an expansion $\langle V_{int}(\lambda) \rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \ldots$, then

$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda} \langle \phi | V_{int}(\lambda) | \phi \rangle = V_1 + \frac{1}{2} V_2 + \frac{1}{3} V_3 + \ldots \tag{20}$$

(d) When we turn on the interaction, the change in the ground-state energy involves the contributions from both the change in the Hamiltonian and the change in the ground-state. The factors of $\frac{1}{n}$ appearing in front of the n-th order terms reflect the fact that the ground-state relaxes in response to the change in hamiltonian, so that the change in the ground-state energy from each term is less than the corresponding change in the expectation value of the interaction.

3. (a) The crosses represent the scattering amplitude $V_{k,k'}$ and the lines represent the propagators.

(b) Diagrammatically, we have:

or

$$G_{\mathbf{k},\mathbf{k}'}(E) = G^{(0)}_{\mathbf{k}}(E) \delta_{\mathbf{k},\mathbf{k}'} + G^{(0)}_{\mathbf{k}}(E) t_{\mathbf{k},\,\mathbf{k}'}(E) G^{(0)}_{\mathbf{k}'}(E) \tag{21}$$

where the "blob" is the t-matrix, represented by the following sum of diagrams

Written algebraically, this becomes

$$t_{\mathbf{k},\mathbf{k}'}(E) = U(\mathbf{k} - \mathbf{k}') + \int \frac{d^d q}{(2\pi)^d} \frac{U(\mathbf{k} - \mathbf{q})}{E - E(q) + i\delta} t_{\mathbf{q},\mathbf{k}'}(E) \tag{22}$$

where we assume that the Fermi surface is empty (i.e $\mu = 0$, so that $\delta_{\mathbf{k}} = \delta$ for all states.)

(c) If $U(x) = U\delta^{(d)}(x)$, then $U(q) = U$ and the t-matrix is now momentum independent. We may immediately solve (22) to obtain

$$t(\omega) = \frac{U}{1 - UF(\omega)} \tag{23}$$

where

$$F(\omega) = \int \frac{d^d k}{(2\pi)^d} G^{(0)}(k, \omega)$$

(d) Let us examine how the integral in the denominator of the t-matrix scales with energy at low energies. We take the case of a "drained Fermi sea", in which the chemical potential $\mu = 0.$ , so that for $\omega < 0$,

$$\begin{aligned} F(\omega) = \int \frac{d^d k}{(2\pi)^d} G^{(0)}(k,\omega) &\propto \int d\epsilon \epsilon^{(\frac{d}{2}-1)} \frac{1}{\omega - \epsilon + i\delta} \\ &\propto -(-\omega)^{(\frac{d}{2}-1)} \\ &\propto -\ln\left(\frac{\Lambda}{-\omega}\right), \qquad (d=2). \end{aligned} \tag{24}$$

Thus in dimensions $d \leq 2$, if $U < 0$, the denominator of the t-matrix

$$1 - UF(\omega) = 1 + |U|F(\omega) \sim 1 - \frac{|U|}{(-\omega)^{(2-d)/2}}$$

will pass through zero at some small $\omega = -\omega^* \propto |U|^{2/(2-d)}$, for arbitrarily small $|U|$, giving rise to a pole in the t-matrix. To see that this means the development of a bound-state, consider the density of one-particle states

$$\rho(\omega) = \sum_\lambda \delta(\omega - E_\lambda) \tag{25}$$

where $E_\lambda$ is the energy of the eigenstate $|\lambda\rangle$. We may rewrite this in the form

$$\begin{aligned} \rho(\omega) &= -\frac{1}{\pi} \mathrm{Im} \sum_\lambda \frac{1}{(\omega - E_\lambda + i\delta)} \\ &= -\frac{1}{\pi} \mathrm{Im} \sum_\lambda \langle \lambda | \hat{G}(\omega) | \lambda \rangle \\ &= -\frac{1}{\pi} \mathrm{Im} \mathrm{Tr} \left[ \hat{G}(\omega) \right], \end{aligned} \tag{26}$$

where $\hat{G}(\omega) = (\omega - H + i\delta)^{-1}$. We may also take the trace by summing over the momentum eigenstates, rather than energy eigenstates, so that

$$\begin{aligned} \rho(\omega) &= -\frac{1}{\pi} \sum_{\mathbf{k}} \mathrm{Im} \langle \mathbf{k} | \hat{G}(\omega) | \mathbf{k} \rangle \\ &= -\frac{1}{\pi} \sum_{\mathbf{k}} \mathrm{Im} G_{\mathbf{k},\mathbf{k}}(\omega) \end{aligned} \tag{27}$$

Writing the Green-function in terms of the t-matrix, the change in the density of states due to scattering is then

$$\Delta\rho(\omega) = -\frac{1}{\pi} \mathrm{Im} \left[ \left( \int \frac{d^d k}{(2\pi)^d} G^{(0)}(k,\omega) \right)^2 t(\omega) \right] \tag{28}$$

Near the pole at negative energies, we may write this in the form

$$\Delta\rho(\omega) = -\frac{1}{\pi} \mathrm{Im}[F(\omega)^2 t(\omega)] = -\frac{1}{\pi} \mathrm{Im} \left[ \left( \frac{1}{U} - \frac{1}{t(\omega)} \right)^2 t(\omega) \right] \approx -\frac{1}{U^2 \pi} \mathrm{Im} \left[ t(\omega) \right] \tag{29}$$

Thus a pole in $t(\omega)$ implies a pole at negative energies in the density of states, indicating a bound-state.