INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025

Questions 4. (Due Mon, Nov 3)

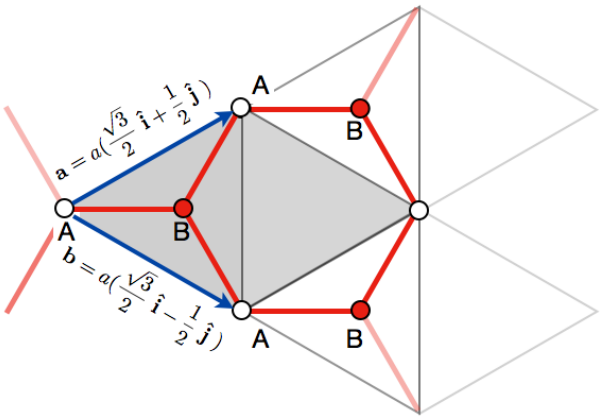


Figure 1: Honeycomb structure of graphene. See Problem 1

1. Electrons on graphene move on a Honeycomb lattice as shown in Fig. (1). The vertices of each unit cell form a triangular lattice of side length a, located at positions $\mathbf{r}_i = m\mathbf{a} + n\mathbf{b}$, where $\mathbf{a} = a\left(\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}\right)$ and $\mathbf{b} = a\left(\frac{\sqrt{3}}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}\right)$ are the lattice vectors. There are two atoms per unit cell, labelled "A" and "B". In a simplified model of graphene, electrons can occupy π orbitals at either the A or the B sites, with a tight-binding hopping matrix element $-t$ between neighboring sites.

   (a) Construct a tight-binding model for graphene. For simplicity, ignore the spin of the electron. Suppose the creation operator for an electron in the A or B orbital in the "i"th cell is $\psi^\dagger{}_\lambda(\mathbf{r}_i)$ $(\lambda = A, B)$. Show that the tight-binding Hamiltonian can be written in the form

$$H = -t\sum_j \left\{\left[\psi^\dagger{}_B(\mathbf{r}_i) + \psi^\dagger{}_B(\mathbf{r}_i - \mathbf{a}) + \psi^\dagger{}_B(\mathbf{r}_i - \mathbf{b})\right]\psi_A(\mathbf{r}_i) + \text{H.c}\right\} + (\epsilon - \mu)\sum_i (n_A(i) + n_B(i))$$

   where ϵ is the energy of a localized orbital.

   (b) By transforming to momentum space, writing

$$\psi^\dagger{}_\lambda(\mathbf{r}_j) = \frac{1}{\sqrt{N_s}}\sum_{\mathbf{k}} c^\dagger{}_{\mathbf{k}\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \qquad (\lambda = A, B)$$

   and N_s is the number of unit cells in the crystal, show that the Hamiltonian can be written

$$H = \sum_{\mathbf{k}} \left(c^\dagger{}_{\mathbf{k}B}, c^\dagger{}_{\mathbf{k}A}\right) \begin{bmatrix} \epsilon - \mu & \Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & \epsilon - \mu \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}B} \\ c_{\mathbf{k}A} \end{pmatrix}$$

where

$$\Delta(\mathbf{k}) = -t(1 + e^{i\mathbf{k}\cdot\mathbf{a}} + e^{i\mathbf{k}\cdot\mathbf{b}})$$

with energy eigenstates

$$\epsilon(\mathbf{k}) = \pm|\Delta(\mathbf{k})| + (\epsilon - \mu).$$

(c) Show that $\Delta(\mathbf{k}) = 0$ at two points in the Brillouin zone where $\mathbf{k}\cdot\mathbf{a} = -\mathbf{k}\cdot\mathbf{b} = \pm\frac{2\pi}{3}$, given by

$$\mathbf{k} = \pm\mathbf{K}$$

where $\mathbf{K} = \frac{4\pi}{3a}\hat{\mathbf{j}}$.

(d) By expanding around $\mathbf{k} = \pm\mathbf{K} + \mathbf{p}$, showing that when $\mathbf{p}$ is small, $\Delta_{\mathbf{p}\pm\mathbf{K}} = \pm\tilde{c}(p_y \pm ip_x)$, where $\tilde{c} = \frac{\sqrt{3}}{2}at$ is a "renormalized" speed of light. By defining a spinor for the two cones

$$\psi_{\mathbf{p}+} = \begin{pmatrix} c_{\mathbf{p}+\mathbf{KB}} \\ c_{\mathbf{p}+\mathbf{KA}} \end{pmatrix}, \qquad \psi_{\mathbf{p}-} = \begin{pmatrix} c_{\mathbf{p}-\mathbf{KA}} \\ -c_{\mathbf{p}-\mathbf{KB}} \end{pmatrix},$$

show that the low energy Hamiltonian can be written as a Dirac equation

$$H = \sum_{\mathbf{p}\lambda=\pm} \psi^\dagger{}_{\mathbf{p}\lambda} \left((\vec{\sigma}\times\mathbf{p}) + (\epsilon-\mu)\underline{\mathbf{1}}\right)\psi_{\mathbf{p}\lambda}$$

where $\vec{\sigma}$ is a Pauli pseudo-spin matrix acting in the two-component sublattice space, so that when $\epsilon - \mu = 0$, the excitation spectrum is defined by two Dirac cones with $E(\mathbf{p}) = \pm\tilde{c}p$.

2. Suppose we turn on the interaction adiabatically, writing $V_{int}(\lambda) = \lambda\hat{V}$. It is possible to relate the change in the Free energy ΔF to the interaction energy averaged over interaction strength with weight $1/\lambda$. Prove this using the following outline.

(a) Consider the Free energy of the system, which is given by $F(\lambda) = -T\ln\mathrm{Tr}[e^{-\beta[H_o+\lambda V]}]$. First show by differentiating this result that

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{\lambda}\langle V_{int}(\lambda)\rangle \tag{1}$$

(b) Use this result to reason that the change in the free energy as a result of turning on the interaction is given by

$$\Delta F = \int_0^1 \frac{d\lambda}{\lambda}\langle V_{int}(\lambda)\rangle \tag{2}$$

By taking the zero temperature limit of this expression, we can reason that

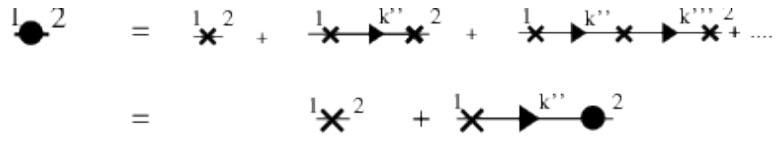
$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda}\langle\phi(\lambda)|V_{int}(\lambda)|\phi(\lambda)\rangle \tag{3}$$

where $|\phi(\lambda)\rangle$ is the ground-state for $H(\lambda) = H_0 + \lambda\hat{V}$. In otherwords, the change in energy is equal to the interaction energy, averaged over the interaction strength, with weight $1/\lambda$.

(c) If the interaction energy has an expansion $\langle V_{int}(\lambda)\rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \ldots$, what is the corresponding expression for the change in the ground-state energy in increasing λ to full strength $\lambda = 1$?

(d) Why is the change in the ground-state energy not equal to the change in the interaction energy? (Hint- as the interaction increases- what is changing, other than the Hamiltonian?)

3. Construct the diagram technique for the one-particle scattering amplitude on the potential $U(x)$.



(a) What is denoted by the crosses and the solid line?

(b) Show that the electron Green's function has the form

$$G_{\vec{k},\vec{k}'}(E) = G^{(0)}_{\vec{k}}(E)\delta_{\vec{k},\vec{k}'} + G^{(0)}_{\vec{k}}(E)t_{\vec{k},\,\vec{k}'}(E)G^{(0)}_{\vec{k}}(E) \tag{4}$$

where the scattering amplitude satisfies the integral equation

$$t_{\vec{k},\vec{k}'}(E) = U(\vec{k}-\vec{k}') + \int \frac{d^dq}{(2\pi)^d}\frac{U(\vec{k}-\vec{q})}{E-E(q)+i\delta}t_{\vec{q},\vec{k}'}(E) \tag{5}$$

(c) Evaluate the t-matrix for a delta-function potential $U(x) = U\delta^{(d)}(x)$. (Hint: the t-matrix is momentum independent).

(d) Show that in dimensions $d \leq 2$, a bound-state forms beneath the bottom of the conduction band for arbitrarily weak attractive scattering potential. (Hint: this means checking to see if there are negative energy poles in the t-matrix)