INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025

Questions 3. (Due Mon, Oct 20th)

1. A novel fluid of fluid of bosons in a cube of size length L is subject to a momentum-diagonal three-body interaction, described by the Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}} \left(\epsilon_{\mathbf{k}} b^{\dagger}{}_{\mathbf{k}} b_{\mathbf{k}} + \frac{U}{3!} (b^{\dagger}{}_{\mathbf{k}})^3 (b_{\mathbf{k}})^3 \right) \tag{1}$$

where $\mathbf{k} = \frac{2\pi}{L}(l, m, n)$ and $\epsilon_{\mathbf{k}} = E_{\mathbf{k}} - \mu$ and $E_{\mathbf{k}} = k^2/2m$ (take $\hbar = 1$.)

   (a) Consider a one-particle momentum state of energy $\epsilon_{\mathbf{k}}$. Enumerate the energies of the corresponding one, two and three particle-states, $|1_{\mathbf{k}}\rangle = b^{\dagger}{}_{\mathbf{k}}|0\rangle$, $|2_{\mathbf{k}}\rangle = \frac{1}{\sqrt{2!}}(b^{\dagger}{}_{\mathbf{k}})^2|0\rangle$ and $|3_{\mathbf{k}}\rangle = \frac{1}{\sqrt{3!}}(b^{\dagger}{}_{\mathbf{k}})^3|0\rangle$. (That is, if $H|n_{\mathbf{k}}\rangle = \mathcal{E}_{n_{\mathbf{k}}}|n_{\mathbf{k}}\rangle$, give the energy eigenvalue $\mathcal{E}_{n_{\mathbf{k}}}$, for $n_{\mathbf{k}} = (1, 2, 3)$.)

   (b) Suppose at low temperatures, U is large enough to ignore states with more than three bosons in any given momentum $\mathbf{k}$, derive an expression for the Free energy that is valid in this case.

   (c) Assuming we can ignore states with more than three bosons in any given momentum $\mathbf{k}$, what is the expectation value of $\langle n_{\mathbf{k}}\rangle$ in thermal equilibrium?

   (d) Plot $\langle n_{\mathbf{k}}\rangle$ as function of $\epsilon_{\mathbf{k}}$ and describe the qualitative features of your result.

2. A Bose Einstein condensate inside an optical atom trap contains alkali atoms at densities of about $10^{14} - 10^{15}\text{cm}^{-3}$.

   (a) What is the Bose Einstein transition temperature of a gas of Sodium atoms at a density 10^{15}cm^{-3}? (Give your answer in micro-Kelvin.) How are such temperatures attained in practice?

   (b) Liquid Helium-4 has a density of 122g/litre at its boiling point. Compare its theoretical Bose Einstein condensation temperature with its superfluid transition temperature (2.21 K). Why are the two numbers not the same?

3. Consider a system of fermions or bosons, created by the field $\psi^{\dagger}(\vec{r})$ interacting under the potential

$$V(r) = \begin{cases} U, & (r < R), \\ 0, & (r > R), \end{cases} \tag{2}$$

   (a) Write the interaction in second quantized form.

   (b) Switch to the momentum basis, where $\psi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} c_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$. Verify that $[c_{\vec{k}},\ c^{\dagger}{}_{\vec{k}'}]_{\pm} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$, and write the interaction in this new basis. Please sketch the form of the interaction in momentum space.