

INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025

Answers to Questions II. Oct 5.

Here is an outline of the solutions.

1. (i) Expanding $|\psi\rangle = |1111100\dots\rangle = c^\dagger_5 c^\dagger_4 c^\dagger_3 c^\dagger_2 c^\dagger_1 |0\rangle$ we obtain (in gory detail)

$$\begin{aligned} c^\dagger_3 c_6 c_4 c_6^\dagger c_3 |\psi\rangle &= c^\dagger_3 c_6 c_4 c_6^\dagger \cancel{c^\dagger_3} c^\dagger_5 c^\dagger_4 \cancel{c^\dagger_3} c^\dagger_2 c^\dagger_1 |0\rangle \\ &= c^\dagger_3 \cancel{c_6} c_4 \cancel{c_6^\dagger} c^\dagger_5 c^\dagger_4 c^\dagger_2 c^\dagger_1 |0\rangle \\ &= -c^\dagger_3 \cancel{c_4} c^\dagger_5 \cancel{c^\dagger_4} c^\dagger_2 c^\dagger_1 |0\rangle \\ &= +c^\dagger_3 c^\dagger_5 c^\dagger_2 c^\dagger_1 |0\rangle \\ &= -c^\dagger_5 c^\dagger_3 c^\dagger_2 c^\dagger_1 |0\rangle = -|1110100\dots\rangle. \end{aligned}$$

- (ii) We may write

$$\begin{aligned} |11010011\dots\rangle &= c^\dagger_8 |11010010\dots\rangle = c^\dagger_8 c^\dagger_7 |11010000\dots\rangle \\ &= c^\dagger_8 c^\dagger_7 c_5 |11011000\dots\rangle \\ &= c^\dagger_8 c^\dagger_7 c_5 c_3 |11111000\dots\rangle \end{aligned} \tag{1}$$

This state can be interpreted as the creation of an electron in states 8 and 7 and a “hole” in states 3 and 5.

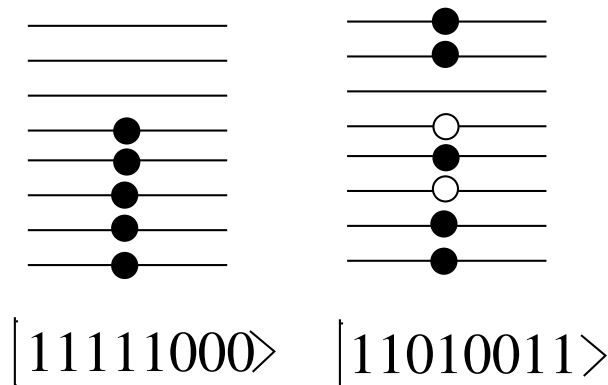


Figure 1:

- (iii) To calculate $\langle \psi | \hat{N} | \psi \rangle$, where $\psi = A|100\dots\rangle + B|111000\dots\rangle$, note that

$$\hat{N}|\psi\rangle = A|100\dots\rangle + 3B|111000\dots\rangle, \tag{2}$$

so that $\langle \psi | \hat{N} | \psi \rangle = |A|^2 + 3|B|^2$.

2. (i) We need to confirm that $\{c_1, c_1^\dagger\} = \{c_2, c_2^\dagger\} = 1$ and also $\{c_1, c_2\} = \{c_2^\dagger, c_1^\dagger\} = 0$. Substituting for c_1 and c_2 , we obtain

$$\{c_1, c_2\} = \{ua_1 + va_2^\dagger, -va_1^\dagger + ua_2\} = -uv\{a_1, a_1^\dagger\} + vu\{a_2^\dagger, a_2\} = 0, \quad (3)$$

and

$$\{c_1, c_1^\dagger\} = \{ua_1 + va_2^\dagger, u^*a_1^\dagger + v^*a_2\} = |u|^2\{a_1, a_1^\dagger\} + |v|^2\{a_2^\dagger, a_2\} = 1, \quad (4)$$

provided $|u|^2 + |v|^2 = 1$.

- (ii) Consider $H = \omega[c_1^\dagger c_1 - c_2 c_2^\dagger]$, then if

$$\begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} = U \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix}, \quad (5)$$

where (assuming u, v real) U is a unitary transformation, we may re-write H as

$$H = (c_1^\dagger, c_2) \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix}$$

so that using (5)

$$\begin{aligned} H &= (a_1^\dagger, a_2) U^\dagger \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} U \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} \\ &= (a_1^\dagger, a_2) \begin{pmatrix} \epsilon & \Delta \\ \Delta & -\epsilon \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} \\ &= \epsilon[a_1^\dagger a_1 - a_2 a_2^\dagger] + [\Delta a_1^\dagger a_2^\dagger + \text{H.c.}] \end{aligned} \quad (6)$$

where

$$\begin{aligned} \epsilon &= \omega(u^2 - v^2), \\ \Delta &= \omega(2uv). \end{aligned} \quad (7)$$

Squaring both expressions and adding the results, we obtain $\omega = (\epsilon^2 + \Delta^2)^{\frac{1}{2}}$ and

$$u^2 = \frac{1}{2} \left(1 + \frac{\epsilon}{\omega}\right), \quad v^2 = \frac{1}{2} \left(1 - \frac{\epsilon}{\omega}\right), \quad (8)$$

- (iii) The ground-state is annihilated by both c_1 and c_2 , so that if $H = \omega[c_1^\dagger c_1 + c_2^\dagger c_2 - 1]$, the ground-state energy is $E_o = -\omega = -(\epsilon^2 + \Delta^2)^{\frac{1}{2}}$.

3. Let us write our starting Hamiltonian in the form

$$H = - \sum_j (J_x S_{j+1}^x S_j^x + J_y S_{j+1}^y S_j^y)$$

$$= -\sum_j \left\{ \frac{J_x + J_y}{4} (S_{j+1}^+ S_j^- + \text{H.c.}) + \frac{J_x - J_y}{4} (S_{j+1}^+ S_j^+ + \text{H.c.}) \right\}, \quad (9)$$

where $S^\pm = S^x \pm iS^y$. Using the Jordan Wigner transformation,

$$\begin{aligned} S_j^z &= (c_j^\dagger c_j - \frac{1}{2}) \\ \mathbf{S}_j^+ &= c_j^\dagger e^{i\pi \sum_{l < j} \hat{n}_l}, \end{aligned} \quad (10)$$

we have

$$\begin{aligned} S_{j+1}^+ S_j^- &= c_{j+1}^\dagger c_j, \\ S_{j+1}^+ S_j^+ &= -c_{j+1}^\dagger c_j^\dagger, \end{aligned} \quad (11)$$

so that

$$H = -\sum_j t [c_{j+1}^\dagger c_j + \text{H.c.}] - \sum_j \Delta [c_{j+1}^\dagger c_j^\dagger + \text{H.c.}] \quad (12)$$

where $t = \frac{J_x + J_y}{4}$, $\Delta = \frac{J_y - J_x}{4}$.

(ii) Transforming to a momentum basis, $c_j^\dagger = \frac{1}{\sqrt{N}} \sum_q d_q^\dagger e^{-ix_j a}$, the Hamiltonian takes the form

$$H = -\sum_{q>0} 2t \cos(qa) [d_q^\dagger d_q - d_{-q} d_{-q}^\dagger] - \sum_q \Delta [e^{-iqa} d_q^\dagger d_{-q}^\dagger + \text{H.c.}] \quad (13)$$

Since $d_q^\dagger d_{-q}^\dagger = -d_{-q}^\dagger d_q^\dagger$ is an odd function of q , we can replace $\Delta e^{-iqa} \rightarrow -i\Delta \sin qa$, to get

$$\begin{aligned} H &= \sum_{q>0} \epsilon_q [d_q^\dagger d_q - d_{-q} d_{-q}^\dagger] + \sum_{q>0} i\Delta_q [d_q^\dagger d_{-q}^\dagger - \text{H.c.}] \\ &= \sum_{q>0} (d_q^\dagger, d_{-q}) \begin{pmatrix} \epsilon_q & i\Delta_q \\ -i\Delta_q & -\epsilon_q \end{pmatrix} \begin{pmatrix} d_q \\ d_{-q}^\dagger \end{pmatrix} \\ &= \sum_{q>0} d_q^\dagger (\epsilon_q \tau_3 - \Delta_q \tau_2) d_q, \end{aligned} \quad (14)$$

where $\epsilon_q = -2t \cos(qa)$, $\Delta_q = 2\Delta \sin qa$ and we have introduced the Nambu notation

$$d_q = \begin{pmatrix} d_q \\ d_{-q}^\dagger \end{pmatrix} \quad (15)$$

Notice how the sum over $q > 0$ is needed so that d_q and d_{-q} are independent. Carrying out the Boguilubov transformation $a_q^\dagger = d_q^\dagger U_q$, or

$$(a_q^\dagger, a_{-q}) = (d_q^\dagger, d_{-q}) \cdot \overbrace{\begin{pmatrix} u_q & -v_q^* \\ v_q & u_q^* \end{pmatrix}}^{U_q}, \quad (16)$$

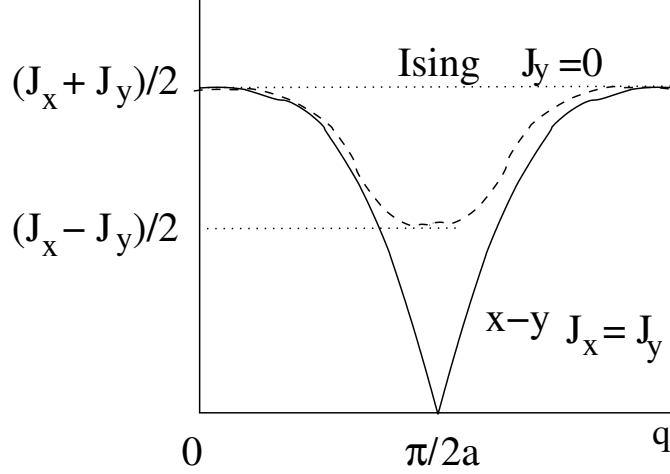


Figure 2: Showing dispersion for x-y, anisotropic x-y and Ising limits of model.

where U_q is the unitary matrix composed of the two column eigenvectors of $\mathcal{H} = \epsilon_q \tau_3 - \Delta_q \tau_2$. Following the results of the last section, the Hamiltonian takes the form

$$H = \sum_{q>0} \omega_q [a_q^\dagger a_q - a_{-q} a_{-q}^\dagger], \quad (17)$$

where

$$\begin{aligned} \omega_q &= \left(\epsilon_q^2 + \Delta_q^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left[J_x^2 + J_y^2 + 2J_x J_y \cos(2qa) \right]^{\frac{1}{2}} \\ \begin{pmatrix} u_q \\ v_q \end{pmatrix} &= \begin{pmatrix} \left[1 + \frac{\epsilon_q}{\omega_q} \right]^{\frac{1}{2}} \\ i \left[1 - \frac{\epsilon_q}{\omega_q} \right]^{\frac{1}{2}} \end{pmatrix}. \end{aligned} \quad (18)$$

The spectrum of spin-excitations is shown above. For the case $J_y = J_x$, the excitation spectrum is gapless, corresponding to the continuous rotational symmetry (Goldstone mode). For the case J_y , or $J_x = 0$, the excitation spectrum is flat $\omega = J_x/2$ as expected for the 1d Ising model. We can interpret $J_x/2$ as the energy to create a “domain wall” in the Ising Ferromagnet. If the ground-state is $|\phi\rangle = |\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rangle$ then the flat band of fermions corresponds to domain wall excitations. If we use open boundary conditions, then energy to create a single domain wall, i.e $|\rightarrow\rightarrow\rightarrow\rightarrow\leftarrow\leftarrow\leftarrow\rangle$ is $J_x/2$. In periodic boundary conditions, such domain walls can only be created in pairs.