

INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025

Questions 2. (Due Weds, Oct 1)

1. In this question c_i^\dagger and c_i are fermion creation and annihilation operators and the states are fermion states. Use the convention

$$|11111000\dots\rangle = c_5^\dagger c_4^\dagger c_3^\dagger c_2^\dagger c_1^\dagger |\text{vacuum}\rangle.$$

(i) Evaluate $c_3^\dagger c_6 c_4 c_6^\dagger c_3 |111110000\dots\rangle$.

(ii) Write $|1101001100\dots\rangle$ in terms of excitations about the “filled Fermi sea” $|1111100000\dots\rangle$. Interpret your answer in terms of electron and hole excitations.

(iii) Find $\langle\psi|\hat{N}|\psi\rangle$ where $|\psi\rangle = A|100\rangle + B|111000\rangle$, $\hat{N} = \sum_i c_i^\dagger c_i$.

2. (a) Consider two fermions, a_1 and a_2 . Show that the Bogoliubov transformation

$$\begin{aligned} c_1 &= u a_1 + v a_2^\dagger \\ c_1^\dagger &= -v a_1 + u a_2^\dagger \end{aligned} \tag{1}$$

where u and v are real, preserves the canonical anti-commutation relations if $u^2 + v^2 = 1$.

(b) Use this result to show that the Hamiltonian

$$H = \epsilon(a_1^\dagger a_1 - a_2 a_2^\dagger) + \Delta(a_1^\dagger a_2^\dagger + \text{H.c.}) \tag{2}$$

can be diagonalized in the form

$$H = \sqrt{\epsilon^2 + \Delta^2}(c_1^\dagger c_1 + c_2^\dagger c_2 - 1) \tag{3}$$

(c) What is the ground-state energy of this Hamiltonian?

3. (i) Use the Jordan Wigner transformation to show that the one dimensional anisotropic XY model

$$H = - \sum_i [J_1 S_x(i) S_x(i+1) + J_2 S_y(i) S_y(i+1)] \tag{4}$$

can be written as

$$H = - \sum_i [t(d_{i+1}^\dagger d_i + \text{H.c.}) + \Delta(d_{i+1}^\dagger d_i^\dagger + \text{H.c.})] \tag{5}$$

where $t = \frac{1}{4}(J_1 + J_2)$ and $\Delta = \frac{1}{4}(J_2 - J_1)$.

(ii) Calculate the excitation spectrum for this model and sketch your results. Comment specifically on the two cases $J_1 = J_2$ and $J_2 = 0$. (Hint: when you Fourier transform the pairing terms, since fermion operators anticommute, the resulting term must be odd in momentum, eliminating the even $\cos k$ part of the exponent, leaving just an odd $\sin k$ function.)