INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2026

Questions I. (Due Fri, Sept 25th.)

These questions can *all* be done with a minimum of algebra. Explain your working. Wherever possible draw pictures and diagrams to illustrate your results. All homeworks to be submitted handwritten on paper.

1. In 1906, in what is arguably the first paper in theoretical condensed matter physics Albert Einstein postulated that vibrational excitations of a solid are quantized with energy $\hbar\omega$, just like the photons in the vacuum. Repeat his calculation for diamond: calculate the energy $E(T)$ of one mole of simple harmonic oscillators with characteristic frequency ω at temperature T and show that the specific heat capacity is

$$C_V(T) = \frac{dE}{dT} = RF\left(\frac{\hbar\omega}{k_B T}\right)$$

where

$$F(x) = \left(\frac{x/2}{\sinh(x/2)}\right)^2.$$

and $R = N_{AV}k_B$ the product of Avagadro's number N_{AV} and Boltzmann's constant k_B. Plot $C(T)$ and show that it deviates from Dulong and Petit's law $C_V = (R/2)$ per quadratic degree of freedom at temperatures $T << \hbar\omega/k_B$.

2. (a) Show that if a is a canonical Bose destruction operator, i.e $[a, a^\dagger] = 1$, the Boguiliubov transformation

$$\begin{aligned} b &= ua + va^\dagger, \\ b^\dagger &= ua^\dagger + va, \end{aligned} \tag{1}$$

(where u and v are real), preserves the canonical commutation relations, provided $u^2 - v^2 = 1$.

(b) Using (a), diagonalize the Hamiltonian

$$H = \omega(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\Delta(a^\dagger a^\dagger + aa), \tag{2}$$

by transforming it into the form $H = \tilde{\omega}(b^\dagger b + \frac{1}{2})$. (Hint: Work backwards from the final Hamiltonian.)

3. This is a heuristic problem on how to scale-up from a harmonic oscillator to the electromagnetic field. We can upgrade from zero to D dimensions by adding momentum and polarization subscripts to the normal modes as follows

Harmonic Oscillator | Field Theory

$$\left.\begin{aligned} H &= \frac{\pi^2}{2m} + \frac{m\omega^2\phi^2}{2} \\ [\phi, \pi] &= i\hbar\ \hat{1} \\ \phi &= \frac{\Delta x}{\sqrt{2}}(a + a^\dagger) \end{aligned}\right\} \quad \xrightarrow[\phi \to \phi_{q\lambda},\ \hat{1} \to \delta_{qq'}\delta_{\lambda\lambda'}\ldots]{\text{Decorate with subscripts}} \quad \left\{\begin{aligned} H &= \sum_{q,\lambda}\left(\frac{|\pi_{q\lambda}|^2}{2m} + \frac{m\omega_q^2|\phi_{q,\lambda}|^2}{2}\right) \\ [\phi_{q\lambda}, \pi^\dagger_{q'\lambda'}] &= i\hbar\delta_{q,q'}\delta_{\lambda\lambda'} \\ \phi_{q\lambda} &= \frac{\Delta x_{q\lambda}}{\sqrt{2}}\left(a_{q\lambda} + a^\dagger_{-q\lambda}\right) \end{aligned}\right.$$

where we denote $\Delta x_q = \sqrt{\hbar/m\omega_q}$, and the modulus sign is shorthand for $|x|^2 = x^\dagger x$. Note that $\phi_{q\lambda}^\dagger = \phi_{-q\lambda}$, $\pi_{q\lambda}^\dagger = \pi_{-q\lambda}$.

(a) For the electromagnetic field, the displacement field is the vector potential, $\phi_{q\lambda} \to A_{q\lambda}$ and the canonical momentum is $\pi_{q\lambda} \to -\epsilon_0 E_{q\lambda}$, where $\lambda = 1, 2$ denotes the polarization of the wave (for the Coulomb gauge where $\nabla \cdot \vec{A} = 0$) Write the well-known field energy

$$H = \int d^3x \left[\frac{\epsilon_0 E^2}{2} + \frac{(\nabla \times \vec{A})^2}{2\mu_0} \right].$$

in momentum space (consider a finite system, with a discrete summation over momenta) and use this to read off m, ω_q and $\Delta x_{q\lambda}$ in terms of ϵ_0, c and q. Here we are basically doing dimnsional analysis. (Hint: $(\nabla \times \vec{A})^2 \to |\vec{q} \otimes \vec{A}_{q\lambda}|^2$)

(b) Write an expression for the normal mode $A_{q\lambda}$ in terms of the photon creation and annihilation operators.

(c) Write out an expression for the normal mode $E_{q\lambda}$ in terms of the photon creation and annihilation operators.

(d) What subtleties have been glossed over?