Solution to Exercise 3, Physics 603. Kitaev Spin Liquid

(Reference: Feng, Zhang, and Xiang, PRL 98, 087204 (2007))

1. The Kitaev honeycomb model can, surprisingly be solved by using an old-fashioned Jordan Wigner transformation. (Jordan, who together with Klein was the inventor of fermion creation and annihilation operators.) To see how this works, first see that we can compress the honeycomb lattice in one direction and redraw it as a brick-wall lattice, composed of one dimensional chains with alternating cross-links (see Fig. 1), where the chains are labelled by the index l , and the position along the chains is labelled by the index j. so that the Hamiltonian can be written

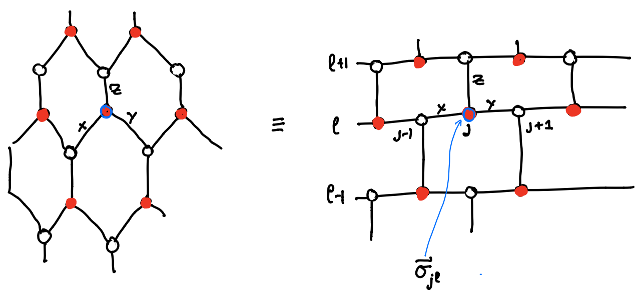

FIG. 1: Equivalence between honeycomb and "brick-wall" lattice.

$$H = -\sum_{l+j\in\text{even}}\left[J^x(\sigma^x_{jl}\sigma^x_{j-1,l}) + J^y(\sigma^y_{jl}\sigma^y_{j+1,l}) + J^z(\sigma^z_{jl}\sigma^z_{j,l+1})\right] \tag{1}$$

(a) Show that the Jordan Wigner transformation can be generalized to each of the horizontal chains through the substitution:

$$\begin{aligned}
\sigma^+_{jl} &= (\sigma^x_{jl} + i\sigma^y_{jl})/2 = f^\dagger_{jl}\exp\left[i\pi\sum_{r<j} n_r\right]\\
\sigma^-_{jl} &= (\sigma^x_{jl} - i\sigma^y_{jl})/2 = f_{jl}\exp\left[-i\pi\sum_{r<j} n_r\right]\\
\sigma^z_{jl} &= 2n_{jl} - 1,
\end{aligned} \tag{2}$$

where $n_r = \sum_l f^\dagger_{rl} f_{rl}$ is the sum over all occupancies in the rth row of spins and f_{rl} is a fermion operator at site (r, l). Verify that the spin operators on different chains commute.

Well. Here I must apologize to you all. I have given you the wrong form for the string operator in this problem. The correct Jordan Wigner transformation has a different string, (see Feng, Zhang, and Xiang, PRL 98, 087204 (2007)) as follows:

$$
\begin{aligned}
\sigma^+_{jl} &= (\sigma^x_{jl} + i\sigma^y_{jl})/2 = f^\dagger_{jl} \exp\left[i\Phi_{jl}\right] \\
\sigma^-_{jl} &= (\sigma^x_{jl} - i\sigma^y_{jl})/2 = f_{jl} \exp\left[-i\Phi_{jl}\right] \\
\sigma^z_{jl} &= 2n_{jl} - 1,
\end{aligned} \tag{3}
$$

where

$$
\Phi_{jl} = \overbrace{\pi \sum_{j'<j} n_{j'l}}^{\text{string on same row}} + \overbrace{\pi \sum_{i,l'<l} n_{il'}}^{\text{continuation of string along lower rows}} \tag{4}
$$

The string operator $\exp[i\Phi_{jl}]$ corresponds to a string that comes in to the left of the site (j,l) having wrapped around all the lower rungs of the lattice as follows:

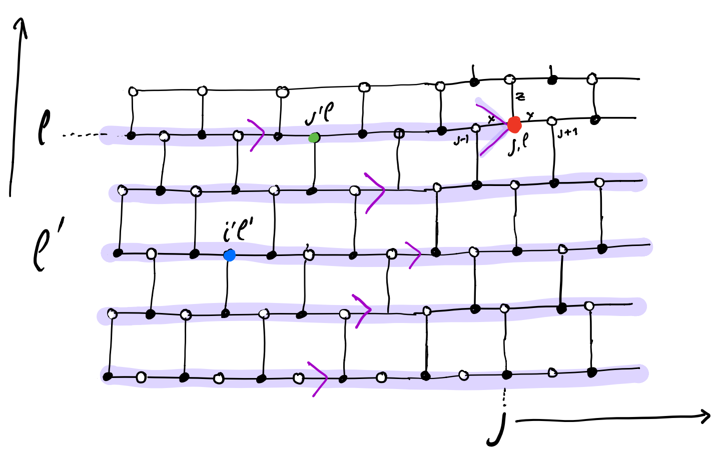

FIG. 2: Showing string $\exp[i\Phi_{jl}]$ snaking backwards and forwards up to site (j,l).

Let us now check that the spins commute between different sites. First, note that because of the 1D string, spins on the same horizontal chain must commute. Explicitly, if $i > j$, then

$$
[\sigma^\pm_{il}, \sigma^\pm_{jl}] = [f^{(\dagger)}_{il} e^{i\pi \sum_{i\le j'\le j-1} n_{j'l}}, f^{(\dagger)}_{jl}] = 0 \tag{5}
$$

But what about spins on different chains? Consider two sites, (i,j) and (l,m) chosen so that chain j is above chain m, $j>m$, then when we calculate the commutator of the raising and lowering operators,

$$\begin{aligned}
[\sigma^{\mp}_{ij}, \sigma^{\pm}_{lm}] &= [f^{(\dagger)}_{ij} e^{-i\Phi_{ij}}, f^{(\dagger)}_{lm} e^{i\Phi_{lm}}] \\
&= [f^{(\dagger)}_{ij} e^{-i(\Phi_{ij}-\Phi_{lm})}, f^{(\dagger)}_{lm}] \\
&= e^{-i\Phi_{ij,lm}}[f^{(\dagger)}_{ij} e^{-i\pi n_{lm}}, f^{(\dagger)}_{lm}] = 0.
\end{aligned} \tag{6}$$

where $\Phi_{ijlm} = \pi \sum_{(p,q)\in\mathcal{A}} n_{pq}$ defines the two dimensional string $\exp[-i\Phi_{ij,lm}]$ that wraps around the successive rungs between the two sites (l,m) and (i,j) (see Fig. 3)

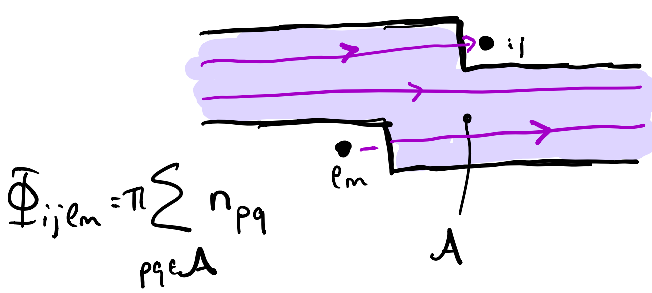

FIG. 3: String running between the two sites (l,m) and (i,j)

(b) Show that the fermionized Hamiltonian can be written in the form

$$H = -\sum_{l+j\in\text{even}} \Big[J^x (f_{jl}+f^\dagger_{jl})(f_{j-1l}-f^\dagger_{j-1l}) + J^y (f^\dagger_{j+1l}-f_{j+1l})(f^\dagger_{jl}+f_{jl}) + J^z (2n_{jl}-1)(2n^z_{j,l+1}-1) \Big]$$

Carrying out the Jordan Wigner transformation, we have

$$\sigma^x_{jl}\sigma^x_{j-1,l} = (f_{j,l}+f^\dagger_{j,l}) e^{i\pi n_{j-1,l}} (f_{j-1,l}+f^\dagger_{j-1,l}) = (f_{j,l}+f^\dagger_{j,l})(f_{j-1,l}-f^\dagger_{j-1,l})$$

and

$$\sigma^y_{j+1,l}\sigma^y_{j,l} = \Big(\frac{f^\dagger_{j+1,l}-f_{j+1,l}}{i}\Big) e^{i\pi n_{j,l}} \Big(\frac{f^\dagger_{j,l}-f_{j,l}}{i}\Big) = (f^\dagger_{j+1,l}-f^\dagger_{j+1,l})(f^\dagger_{j,l}+f_{j,l})$$

while

$$\sigma^z_{jl}\sigma^z_{j,l+1} = (2n_{jl}-1)(2n_{j,l+1}-1)$$

so that the fermionized Hamiltonian becomes

$$H = -\sum_{l+j\in\text{even}}\left[J^x(f_{jl}+f^\dagger_{jl})(f_{j-1l}-f^\dagger_{j-1l})+J^y(f^\dagger_{j+1l}-f_{j+1l})(f^\dagger_{jl}+f_{jl})+J^z(2n_{jl}-1)(2n^z_{j,l+1}-1)\right]$$

(c) Split the fermions into their Majorana components, writing $f_{jl} = (c_{jl} + ib_{jl})/2$ ($j + l$ even) and $f_{jl} = (b_{jl} - ic_{jl})/2$ ($j + l$ odd), (where $c^2_{jl} = 1$) to show that

$$H = \frac{1}{2}\sum_{j+l\in\text{even}}[i(J^x c_{j-1l} + J^y c_{j+1l} + J^z c_{jl+1}u_{jl+1,jl})c_{jl} + \text{H.c}] \tag{7}$$

where $u_{jl+1,jl} = -ib_{jl+1}b_{jl}$ is a Z_2 field that lives on the vertical z bonds. You should show that $\hat{u}_{jl+1,jl} = \pm 1$ commutes with the Hamiltonian.

Splitting the fermions into their Majorana components, then for even $j+1$, $f_{jl} = (c_{jl}+ib_{jl})/2$, while $f_{j\pm 1l} = (b_{j\pm 1l}-ic_{j\pm 1,l})/2$ so that $f_{jl}+f^\dagger_{jl} = c_{jl}$ and $f_{j\pm 1,l} - f^\dagger_{j\pm 1,l} = -ic_{j\pm 1,l}$, while

$$\begin{aligned}(2n_{jl} - 1)(2n^z_{j,l+1} - 1) &= (-ib_{jl}c_{jl})(-ib_{j,l+1}c_{j,l+1}) = (ic_{j,l+1}c_{jl})\overbrace{(ib_{jl}b_{j,l+1})}^{u_{jl+1,jl}}\\ &= ic_{j,l+1}u_{jl+1,jl}c_{jl}\end{aligned} \tag{8}$$

from which it follows that

$$H = \frac{1}{2}\sum_{j+l\in\text{even}}[i(J^x c_{j-1l} + J^y c_{j+1l} + J^z c_{jl+1}u_{jl+1,jl})c_{jl} + \text{H.c}]\,. \tag{9}$$

Notice that the operator $u_{jl+1,jl} = ib_{jl}b_{j,l+1}$ commutes with all the $c-$ Majoranas, and only appears in one Ising vertical bond in the Hamiltonian, so it commutes with the entire Hamiltonian and can be treated as a c-number constant of motion.

(d) What is the relationship between this model, and the Kitaev model, written in terms of Majorana fermions?

This model corresponds to a Kitaev model in which the gauge fields along the x- and y- bonds have been set to unity. With this gauge choice the flux through each hexagon is determined by the vertical bonds alone.

(e) What is the spectrum of Majorana excitations in the ground-state?

Fourier transforming the Majorana fermions, writing

$$c_{jl} = \sqrt{\frac{2}{N}}\sum_{\mathbf{k}} c_{\mathbf{Ik}}e^{i\mathbf{k}\cdot\mathbf{R}_{jl}}$$

and

$$c_{j\pm 1l} = \sqrt{\frac{2}{N}} \sum_{\mathbf{k}} c_{\mathrm{II}\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_{j\pm 1l}}$$

for even $j+l$, then setting the gauge fields to unity, the Fourier transformed Hamiltonian becoes

$$\begin{aligned} H &= \frac{1}{2}\sum_{\mathbf{k}}\left[c^{\dagger}_{\mathrm{II}\mathbf{k}}(i2J^{x}e^{-i\mathbf{k}\cdot(\mathbf{R}_{j-1l}-\mathbf{R}_{jl})} + i2J^{y}e^{-i\mathbf{k}\cdot(\mathbf{R}_{j+1l}-\mathbf{R}_{jl})} + 2J_{z})c_{\mathrm{I}\mathbf{k}} + \mathrm{H.c}\right] \\ &= \frac{1}{2}\sum_{\mathbf{k}}\left[c^{\dagger}_{\mathrm{II}\mathbf{k}}(i2J^{x}e^{i\mathbf{k}\cdot\mathbf{n}_1} + i2J^{y}e^{i\mathbf{k}\cdot\mathbf{n}_2} + 2J_{z})c_{\mathrm{I}\mathbf{k}} + \mathrm{H.c}\right] \\ &= \frac{1}{2}\sum_{\mathbf{k}}\left(c^{\dagger}_{\mathrm{I}\mathbf{k}},\ c^{\dagger}_{\mathrm{II}\mathbf{k}}\right)\begin{pmatrix} 0 & -if^{*}_{\mathbf{k}} \\ if_{\mathbf{k}} & 0 \end{pmatrix}\begin{pmatrix} c_{\mathrm{I}\mathbf{k}} \\ c_{\mathrm{II}\mathbf{k}} \end{pmatrix} \end{aligned} \tag{10}$$

where $\mathbf{n}_1 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$ and $\mathbf{n}_2 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$, while

$$f_{\mathbf{k}} = (2J_{x}e^{i\mathbf{k}\cdot\mathbf{n}_1} + 2J^{y}e^{i\mathbf{k}\cdot\mathbf{n}_2} + 2J_{z}). \tag{11}$$

the corresponding spectrum is then $E_{\mathbf{k}} = \pm|f_{\mathbf{k}}|$, corresponding to the Dirac spectrum of graphene.

(f) What excitation (s) is/are created by flipping the sign of $u_{jl+1,jl}$ and what is the approximate energy of the resulting state?

When we flip the sign of one vertical bond, we create two vizons. From perturbation theory around the toric code, we know that the approximate energy of two visons is

$$\Delta E = 4J_{e} = \frac{J_x^2 J_y^2}{4J_z^3}.$$