**GRADUATE QUANTUM MECHANICS: 501 Fall 2001**

**Midterm exam: Solutions.**

1. (a) For any two observables, $\hat{A}$ and $\hat{B}$, the Heisenberg uncertainty principle gives

$$\Delta A \Delta B \geq |\frac{-i}{2}\langle [A, B]\rangle| \tag{1}$$

where $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$, $\Delta B = \sqrt{\langle B^2\rangle - \langle B\rangle^2}$. In this case, if $A = S_x$ and $B = S_y$, then $\frac{-i}{2}[S_x, S_y] = \frac{\hbar}{2}S_z$ so that

$$\Delta S_x \Delta S_y \geq |\frac{\hbar}{2}\langle S_z\rangle| \tag{2}$$

Notice how the uncertainties in $S_x$ and $S_y$ are maximal in states that are fully polarized along the z-axis.

(b) In this problem, if $d$ is the distance over which the virtual particle travels, then the lifetime of the particle is

$$\Delta t \sim \frac{d}{c} \tag{3}$$

The uncertainty in energy is governed by the energy-time uncertainty relation

$$\Delta t \Delta E \geq \frac{\hbar}{2} \tag{4}$$

and since $\Delta E \sim mc^2$, where $m$ is the mass of the particle, it follows that

$$mc^2 \sim \Delta E \sim \frac{\hbar}{2\Delta t} = \frac{\hbar c}{2d} \tag{5}$$

so that

$$m \sim \frac{\hbar}{2dc} \sim \frac{10^{-34}}{2 \times 10^{-6} \times 3 \times 10^{8}} \approx 10^{-37}\text{kg} \tag{6}$$

so that the particle is $m/m_e \sim 10^{-7}$ times less massive than an electron.

(c) This problem was identical to our treatment of spatial translation in class. If we compute the matrix element of the angular translation operator between the state $|\phi\rangle$ localized at $\phi$ and a general state $|\psi\rangle$ we obtain

$$\langle\phi|T_{d\phi}|\psi\rangle = \langle\phi - d\phi|\psi\rangle = \psi(\phi - d\phi) = \psi(\phi) - d\phi\frac{d\psi}{d\phi}, \tag{7}$$

But since $T_{d\phi} = 1 - i\Lambda d\phi$, we can also write

$$\langle\phi|T_{d\phi}|\psi\rangle = \langle\phi|\psi\rangle - id\phi\langle\phi|\Lambda|\psi\rangle, \tag{8}$$

Comparing (7) with (8), we obtain

$$\langle\phi|\Lambda|\psi\rangle = -i\frac{d\psi(\phi)}{d\phi}. \tag{9}$$

(d) The important points are that

i. the amplitude is roughly proportional to $1/\sqrt{p(x)}$, so it is largest where $V(x)$ is smallest! Thank you for your enlightened discussion on this point.

ii. the wavelength is shortest where $V(x)$ is the smallest.

iii. $\psi(x) = 0$ for $x \leq 0$.

iv. The three lowest states, represented by wavefunctions $\psi_0(x)$, $\psi_1(x)$ and $\psi_2(x)$, have zero, one and two nodes respectively.

v. The curvature of the wavefunction changes sign as one crosses from the classically allowed, to the classically forbidden region.

Sketching the key results, we have:

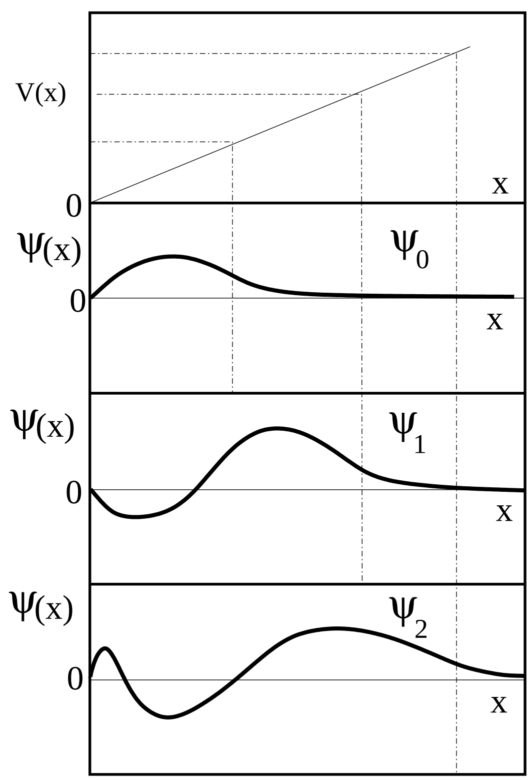


Figure 1: Sketch diagram of wavefunctions of lowest three states of linear potential.

2. (a) For this problem, Schrödinger's equation, $i\hbar\frac{\partial}{\partial t}\psi = H\psi$ becomes

$$\frac{\partial}{\partial t}\psi = \frac{-i}{\hbar}H\psi = i\left[\frac{2eB}{\hbar}\right]S_x\psi, \tag{10}$$

which can be written in the z basis as :

$$\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = i\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{11}$$

where $\omega = eB/m$. The general solution is then

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = A \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} e^{i\omega t} + B \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} e^{-i\omega t} \tag{12}$$

Now since $\psi_1(0) = \psi_1$, $\psi_2(0) = \psi_2$, it follows that $A = \frac{\psi_1+\psi_2}{2}$, $B = \frac{\psi_1-\psi_2}{2}$ and so

$$\begin{aligned} \psi_1(t) &= \psi_1 \cos\omega t + i\psi_2 \sin\omega t, \\ \psi_2(t) &= \psi_2 \cos\omega t + i\psi_1 \sin\omega t. \end{aligned} \tag{13}$$

(b) The Heisenberg equations of motion for the components of $\vec{S} = (S_x, S_y, S_z)$ are

$$\frac{d\vec{S}(t)}{dt} = \frac{i}{\hbar}[H, \vec{S}(t)] = -i\frac{2\omega}{\hbar}[S_x(t), \vec{S}(t)] \tag{14}$$

so that

$$\begin{aligned} \frac{d\vec{S}_x(t)}{dt} &= 0 \\ \frac{d\vec{S}_y(t)}{dt} &= -i\frac{2\omega}{\hbar}[S_x(t), S_y(t)] = 2\omega S_z(t) \\ \frac{d\vec{S}_z(t)}{dt} &= -i\frac{2\omega}{\hbar}[S_x(t), S_z(t)] = -2\omega S_y(t) \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} S_x(t) &= S_x(0) \\ S_y(t) &= S_y(0)\cos 2\omega t + S_z(0)\sin 2\omega t \\ S_z(t) &= S_z(0)\cos 2\omega t - S_y(0)\sin 2\omega t. \end{aligned} \tag{16}$$

corresponding to a spin precessing about the $x$ axis with angular velocity $2\omega$.

(c) The two quantities entering into $\Delta S_z(t)^2 = \langle S_z(t)^2\rangle - \langle S_z(t)\rangle^2$ are

$$\langle\psi|S_z^2(t)|\psi\rangle = \frac{\hbar^2}{4}(|\psi_+|^2 + |\psi_-|^2) = \frac{\hbar^2}{4} \tag{17}$$

and

$$\langle\psi|S_z(t)|\psi\rangle = \cos 2\omega t\langle\psi|S_z(0)|\psi\rangle - \sin 2\omega t\langle\psi|S_y(0)|\psi\rangle = \frac{\hbar}{2}\cos(2\omega t) \tag{18}$$

so that

$$\Delta S_z^2(t) = \langle\psi|S_z^2(t)|\psi\rangle - \langle\psi|S_z(t)|\psi\rangle^2 = \frac{\hbar^2}{4}\sin^2 2\omega t. \tag{19}$$

3. There was a small misprint in this question, with the definition of the creation and annihilation operators, which should have read $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - i\frac{1}{m\omega}p)$ and $a = \sqrt{\frac{m\omega}{2\hbar}}(x + i\frac{1}{m\omega}p)$.

(a) Writing $\frac{x}{\Delta x} = (a+a^\dagger)/\sqrt{2}$, where $\Delta x = \sqrt{\frac{\hbar}{m\omega}}$, we may replace the two terms in the Hamiltonian as follows

$$\begin{aligned} H &= \overbrace{\left[\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}\right]}^{\hbar\omega(a^\dagger a+\frac{1}{2})} - \overbrace{F\hat{x}}^{F\Delta x(a+a^\dagger)/\sqrt{2}} \\ &= \hbar\omega(a^\dagger a + \frac{1}{2}) - \frac{F\Delta x}{\sqrt{2}}(a+a^\dagger) \end{aligned} \tag{20}$$

(b) If we write $\hat{b} = \hat{a} - c$, $\hat{b}^\dagger = \hat{a}^\dagger - c^*$, then since the numbers $c$ and $c^*$ commute with the operators $b$ and $b^\dagger$,

$$[b, b^\dagger] = [a-c, a^\dagger - c^*] = [a, a^\dagger] - [c, a^\dagger] - [a, c^*] + [c.c^*] = [a, a^\dagger] = 1 \tag{21}$$

satisifes the canonical commutation relations.

(c) Substituting $a = b + c$, $a^\dagger = b^\dagger + c^*$ into the Hamiltonian, we obtain

$$\begin{aligned} H &= \hbar\omega[(b^\dagger + c^*)(b+c) + \frac{1}{2}] - \frac{F\Delta x}{\sqrt{2}}(b + b^\dagger + c + c^*) \\ &= \hbar\omega(b^\dagger b + \frac{1}{2}) + \left[\left(\hbar\omega c^* - \frac{F\Delta x}{\sqrt{2}}\right) b + \text{H.c.}\right] + \hbar\omega c^* c - \frac{F\Delta x}{\sqrt{2}}(c + c^*) \end{aligned} \tag{22}$$

Choosing

$$c^*\hbar\omega = c\hbar\omega = \frac{F\Delta x}{\sqrt{2}} \tag{23}$$

then

$$H = \hbar\omega(b^\dagger b + \frac{1}{2}) - \frac{F^2}{2m\omega^2} \tag{24}$$

so that the Hamiltonian can be cast in the canonical form $H = Ab^\dagger b + B$, where

$$A = \hbar\omega, \;\; B = \frac{\hbar\omega}{2} - \frac{F^2}{2m\omega^2} \tag{25}$$

(d) The ground-state is clearly the state which is annihilated by $b$, $b|0\rangle = 0$. In this state, the energy is given by $E_g = B$, or

$$E_g = \frac{\hbar\omega}{2} - \frac{F^2}{2m\omega^2} \tag{26}$$

The expectation value of position is given by

$$\langle x \rangle = \frac{\Delta x}{\sqrt{2}}\langle 0|\left[\overbrace{(b+b^\dagger)}^{\to 0} + (c+c^*)\right]|0\rangle = \frac{\Delta x}{\sqrt{2}}(c+c^*) = \frac{F}{m\omega^2} \tag{27}$$

Classically, we would expect that a displacement $\Delta x = F/k = F/(m\omega^2)$, so the classical and quantum results coincide in this case. The classical ground-state energy would be $E_{cl} = -\frac{F\Delta x}{2} = -\frac{F^2}{2m\omega^2}$. The quantum ground-state energy $\hbar\omega/2$ larger than the classical energy $E_{cl}$, due to the quantum zero-point motion.

(e) The eigenket of the n-th excited state of the system is given by

$$|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{(a^\dagger - c)^n}{\sqrt{n!}}|0\rangle \tag{28}$$

where $|0\rangle$ is the vacuum that is annihilated by the $b$ operator. The corresponding energy is

$$E_n = \hbar\omega(n + \frac{1}{2}) - \frac{F^2}{2m\omega^2} \tag{29}$$