GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Midterm exam, Oct 23rd. Due Oct 30th.

1. Short questions.
   (a) What is the Heisenberg uncertainty relation between the uncertainty ΔS_x and ΔS_y in the x and y component of the spin of a particle?
   (b) A team of experimentalists discovers a relativistic particle that transmits a a new force over a distance of 1 micron. Use the energy-time uncertainty principle to compare its mass with that of an electron.
   (c) A particle is constrained to move on a circular ring: $|\phi\rangle$ describes the state localized at an angle ϕ along the ring and $T_{d\phi} = 1 - i\Lambda d\phi$ is the infinitesimal angular translation operator, so that $T_{d\phi}|\phi\rangle = |\phi + d\phi\rangle$. If $\langle\phi|\psi\rangle = \psi(\phi)$ is the wavefunction of a particle, what is $\langle\phi|\Lambda|\psi\rangle$?
   (d) Sketch the wavefunction of the lowest three eigenstates of a particle in a potential

$$V(x) = \begin{cases} kx & (x > 0) \\ \infty & (x < 0) \end{cases} \tag{1}$$

   taking care to show how the wavelength and amplitude vary with position.

   Note: $[S_x, S_y] = i\hbar S_z$, $\hbar = 1 \times 10^{-34} Js$, $c = 3 \times 10^8 m/s$, $m_e = 9.1 \times 10^{-31}$kg. .

2. A spin 1/2 is subject to a constant magnetic field B in the x direction, giving rise to a Hamiltonian

$$H = -\frac{2eB}{m} S_x \tag{2}$$

   (a) If a spin wavefunction is starts out in the state $|\psi\rangle = |+\rangle\psi_1 + |-\rangle\psi_2$, where $|\pm\rangle$ are the eigenkets of the S_z operator, use Schrödinger's equation to compute the time evolution of $\psi_1(t)$ and $\psi_2(t)$.
   (b) Calculate the equation of motion of the spin operators $S_x(t),\ S_y(t),\ S_z(t)$ in the Heisenberg representation.
   (c) If initially the spin is pointing in the $+z$ direction, how does the uncertainty in S_z, $\langle S_z^2(t)\rangle - \langle S_z(t)\rangle^2$ vary with time?

3. A harmonic oscillator is subject to a constant force F, so that the Hamiltonian becomes

$$H = \left[\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}\right] - F\hat{x}. \tag{3}$$

   (a) Rewrite this Hamiltonian in terms of the creation and annihilation operators $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x + i\frac{1}{m\omega}p)$ and $a = \sqrt{\frac{m\omega}{2\hbar}}(x - i\frac{1}{m\omega}p)$.
   (b) Show that if one shifts the creation and annihilation operators by a constant c so that $\hat{b} = \hat{a} - c$, $\hat{b}^\dagger = \hat{a}^\dagger - c^*$, then b and $b^\dagger$ still satisfy the canonical commutation algebra, $[b, b^\dagger] = 1$.
   (c) By adjusting c, show that it is possible to cast the Hamiltonian in standard form $H = Ab^\dagger b + B$, where you should find both A and B.
   (d) Calculate the ground-state energy, and the expectation value $< x >$ in the ground-state. How does your result compare with that expected classically?
   (e) Write down an expression for the eigenket of the n-th excited state of the system and give the corresponding energy.