GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Assignment 8.

Read Shankar Chapter 13.

1. A spin $\frac{1}{2}$ is initially in the state $|z,+\rangle$. The spin is rotated (i) through angle $\pi/2$ about the y axis, then (ii) through angle $\pi/2$ about the z axis, and finally (iii) through angle $\pi/2$ about the x axis. What is state after each of these three stages? Does the spin return to precisely the initial state?

2. An ensemble of electrons have spins which point 50% in the $+y$ direction and 50% in the $+z$ direction.

   (a) what is the spin density matrix for this ensemble?

   (b) What are the expectation values for the three components of the spin $(\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$?

   (c) If a magnetic field of strength B is applied in the $+z$ direction, what is the spin density matrix after time t?

3. In the state $|jm\rangle$, $J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle$ and $J_z|jm\rangle = \hbar m|jm\rangle$.

   (a) Show that in this state, $\langle J_x\rangle = \langle J_y\rangle = 0$.

   (b) Show that in this state
$$\langle J_x^2\rangle = \langle J_y^2\rangle = \frac{1}{2}\hbar^2[j(j+1) - m^2]$$

   (c) Show that ΔJ_x and ΔJ_y are consistent with Heisenberg's uncertainty relation for angular momentum.

   (d) Show that the states $|j, \pm j\rangle$ are minimum uncertainty wavepackets.

4. A particle with angular momentum $l = 0$ moves in an attractive central potential field with potential given by
$$U(r) = \begin{cases} -U_0 & r < R \\ 0 & r > R \end{cases}$$
where $U_0 > 0$.

   (a) Construct the bound-state wavefunctions.

   (b) By matching the *logarithmic derivative* of the radial wavefunction at $r = R$, show that the binding energy B of each bound-state (where $B = -E > 0$ is the negative of the energy eigenvalue of the bound-state) must satisfy:
$$\begin{aligned} \kappa\cot(\kappa R) &= -\sqrt{\frac{2mB}{\hbar^2}} \\ \kappa &= \sqrt{\frac{2m(U-B)}{\hbar^2}} \end{aligned}$$

   (c) What is the smallest value of U_0 (i.e the shallowest potential well) for which a single bound-state will form?