GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Assignment 7. (Due Mon, 26th)

Read Sakurai, 169-191.

1. (a) Show that an arbitrary two-dimensional matrix M can be written

$$M = m_0 \underline{1} + \sum_{a=1,3} m_a \sigma_a$$

where

$$m_0 = \frac{1}{2}\mathrm{Tr}[M], \quad m_a = \frac{1}{2}\mathrm{Tr}[M\sigma_a]$$

(b) If $\hat{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ is a unit vector, calculate the eigenvectors and eigenvalues of the matrix $\hat{n}\cdot\sigma$.

(c) A spin $\frac{1}{2}$ is rotated through 180^0 about the y axis. The state vector transforms from $|\alpha\rangle$ to $|\alpha\rangle_R = U_y(\pi)|\alpha\rangle$. Give the explicit matrix representation of the rotation operator $U_y(\pi)$.

2. (Sakurai, ch 2, problem 19, p 147.) Let

$$J_\pm = \hbar a^\dagger_\pm a_\mp, \qquad J_z = \frac{\hbar}{2}(a^\dagger_+ a_+ - a^\dagger_- a_-), \qquad 2S = a^\dagger_+ a_+ + a^\dagger_- a_-$$

where $a_\pm$ and $a^\dagger_\pm$ are the annihilation and creation operators for a "spin up" and "spin down" boson, which behave as two *independent* simple harmonic oscilltors, satisfying the canonical commutation relations, $[a_\sigma, a^\dagger_{\sigma'}] = \delta_{\sigma\sigma'}$. Prove that

$$[J_z, J_\pm] = \pm\hbar J_\pm, \qquad [\mathbf{J}^2, J_z] = 0, \qquad \mathbf{J}^2 = \hbar^2 S(S+1)$$

In otherwords, that $2S$ spin bosons automatically combine to create a single spin S. These bosons are called "Schwinger Bosons".

3. (Sakurai, Problem 35, p 150 Chapter 2.) Consider the Hamiltonian of a spinless particle of charge q. In the presence of a static magnetic field,

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A}$$

where $\mathbf{A}$ is the appropriate vector potential. Suppose for simplicity that the magnetic field is in the $+z$ direction. Show, by expanding the Hamiltonian, that the presence of the magnetic field leads to two new interaction terms. Show that the term linear in the magnetic field leads to the correct expression for the interaction of the orbital magnetic moment $(q/2m)\mathbf{L}$ with the magnetic field. Show that there is also an extra term proportional to $B^2(x^2+y^2)$, and comment briefly on its physical significance.