GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Solutions to Assignment 3.

1. (a) Since $U|a^r\rangle = |b^r\rangle = \sum_s |a^s\rangle U_{sr}$, by writing the transformation as

$$(U|+\rangle, U|-\rangle) \quad = \quad (|+\rangle, |-\rangle)\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{1}$$

we can read off the matrix elements of U to be

$$[\hat{U}]_{sr} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}. \tag{2}$$

(b) Under this transformation,

$$|y;\pm\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{i}{\sqrt{2}} \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{i}{\sqrt{2}} \end{pmatrix} = e^{\mp i\frac{\theta}{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{i}{\sqrt{2}} \end{pmatrix} \equiv e^{\mp i\frac{\theta}{2}}|y;\pm\rangle, \tag{3}$$

so that

$$\hat{|}y;\pm\rangle = e^{\mp i\frac{\theta}{2}}|y;\pm\rangle. \tag{4}$$

(c) Since $H = -\frac{eB}{m}S_y$, $H|y;\pm\rangle = \pm\frac{\hbar\omega_c}{2}|y;\pm\rangle$, where $\omega_c = \frac{|e|B}{m}$, so that the time evolution of these states is given by

$$|y;\pm\rangle \rightarrow e^{-i\hat{H}t/\hbar}|y;\pm\rangle = e^{-i\frac{\omega_c t}{2}}|y;\pm\rangle, \tag{5}$$

permitting us to identify $\theta = \omega_c t$.

(d) The precession angle of the spin is given by $\theta = \omega_c t$. If $\theta = 90^0 \equiv \pi/2$, then the time to rotate through 90^0 is

$$t = \left(\frac{\pi}{2}\frac{m}{eB}\right) = \left(\frac{\pi \times 9.1\times 10^{-31}\mathrm{kg}}{2\times 1.6\times 10^{-19}\mathrm{C}\times 1\mathrm{Tesla}}\right) = 8.9\times 10^{-12}\mathrm{s} \tag{6}$$

2. Since $\psi(x) = \delta(x - x_0)$, it follows that the momentum space wavefunction is

$$\phi(p) = \langle p|\psi\rangle = \int_{-\infty}^{\infty} dx\langle p|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi\hbar}}e^{-i\frac{px}{\hbar}}\delta(x-x_0) = \frac{1}{\sqrt{2\pi\hbar}}e^{-i\frac{px_0}{\hbar}}. \tag{7}$$

(a) The time-dependent momentum space wavefunction is then given by

$$\phi(p,t) = \langle p|e^{-iHt/\hbar}|\psi\rangle = e^{-i\frac{p^2 t}{2m}\frac{1}{\hbar}}\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-i\left(px_o + \frac{p^2 t}{2m}\right)\frac{1}{\hbar}}. \tag{8}$$

(b) Transforming back to real space, we have

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty} dp\langle x|p\rangle\phi(p,t) \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{i\left(p(x-x_o)-\frac{p^2 t}{2m}\right)\frac{1}{\hbar}}\end{aligned} \tag{9}$$

Using the result

$$\int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2+bp} = \sqrt{\frac{2\pi}{a}} \exp\left[\frac{b^2}{2a}\right], \tag{10}$$

putting $a = \frac{it}{m\hbar}$ and $b = i\frac{x-x_0}{\hbar}$, we obtain

$$\text{Amplitude}(x_o \to x, \Delta t) \equiv \psi(x,\Delta t) = \sqrt{\frac{m}{ih\Delta t}} \exp\left[\frac{iS}{\hbar}\right] \tag{11}$$

where

$$S = \frac{m}{2}\left(\frac{x-x_o}{\Delta t}\right)^2 \Delta t \tag{12}$$

is the classical action $S = \int_0^t dt' \text{ K.E.}(t')$ for a free particle travelling from x_o to x.

3. (a) The Hamiltonian of the simple Harmonic oscillator is

$$H = \hbar\omega[a^\dagger a + \frac{1}{2}] \tag{13}$$

where a and $a^\dagger$ satisfy the algebra $[a, a^\dagger] = 1$. Physically, $a^\dagger$ creates a single "phonon" of energy $\hbar\omega$. The quantity $\hat{N} = a^\dagger a$ is the number operator, which satisfies $[N, a] = [a^\dagger, a]a = -a$, so that $[a, H] = -\hbar\omega[N, a] = \hbar\omega a$ and the Heisenberg equation of motion for $a(t)$ is

$$\frac{da(t)}{dt} = \frac{1}{i\hbar}[a(t), H] = -i\omega a(t) \tag{14}$$

which we can integrate to obtain $a(t) = e^{-i\omega t}a$.

(b) The n-th excited state $|n\rangle$ is obtained by acting on the ground-state n times with the creation operator $a^\dagger$,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{15}$$

where the pre-factor is introduced to normalize the state.

(c) We can write $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle + \langle|a\rangle]$. Now the position operator x can be written as

$$x = \Delta x[a + a^\dagger], \qquad \Delta x = \sqrt{\frac{\hbar}{2m\omega}} \tag{16}$$

In the Heisenberg representation, this becomes

$$\begin{aligned} x(t) &= \Delta x[a(t) + a^\dagger(t)] \\ &= \Delta x[ae^{-i\omega t} + a^\dagger e^{i\omega t}] \end{aligned} \tag{17}$$

To calculate the time dependent expectation value of position, we simply calculate the expectation value of $x(t)$ in the state $|\psi\rangle$, which is

$$\langle x(t)\rangle = \langle\psi|\hat{x}(t)|\psi\rangle = \frac{\Delta x}{2}\left(\langle 0| + \langle 1|\right)[ae^{-i\omega t} + a^\dagger e^{i\omega t}]\left(|0\rangle + |1\rangle\right) \tag{18}$$

Now only the cross-terms $\langle 0|a|1\rangle = \langle 1|a^\dagger|0\rangle = 1$ survive, so that

$$\langle x(t)\rangle = \Delta x \cos(\omega t) = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) \tag{19}$$

so in the mixed state $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ the expectation value of the position operator oscillates like a cosine wave.

(d) The experimentalist's results are consistent with the absorption of an odd number of photons, with frequency ω. This will then put the system in the n-th excited state. But if n is odd, the wavefunction of the system is an odd-function of position, vanishing at the origin, so that in this excited state, the electron is never found at the origin. We say that this excited state is "odd-parity" because it is odd under the reflection operator. Physically, the photon is an odd-parity particle, and this is why the absorption of odd number of photons leads to an odd-parity electron state.