GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Solution to assignment 2.

1. (a) We begin by writing the eigenvalue equation in the form

$$A_{ij}\psi_j = a\psi_j \tag{1}$$

or

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = a \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{2}$$

The eigenvalues are then given by $\det[a\underline{1} - A] = a(a^2 - 1) = 0$, so that the eigenvalues are $(a^+, a^{(0)}, a^-) = (1, 0, -1)$. By inspection, the corresponding eigenvectors are

$$\begin{aligned} \psi^{(+)} &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \underline{A}\psi^+ = \psi^+ \\ \psi^{(0)} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \underline{A}\psi^0 = \psi^0 \\ \psi^{(-)} &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \underline{A}\psi^- = -\psi^- \end{aligned} \tag{3}$$

(b) Taking matrix elements of $\hat{B}$, we get

$$\begin{aligned} B_{ij} &= 3\langle i|-\rangle\langle -|j\rangle + 2\langle i|0\rangle\langle 0|j\rangle + \langle i|+\rangle\langle +|j\rangle \\ &\equiv [3\psi^{(-)} \otimes \psi^{\dagger(-)} + 2\psi^{(0)} \otimes \psi^{\dagger(0)} + 1\psi^{(+)} \otimes \psi^{\dagger(+)}]_{ij} \\ &= 3\begin{pmatrix} 1/2 & 0 & -1/2 \\ & & \\ -1/2 & 0 & 1/2 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 & 1/2 \\ & & \\ 1/2 & 0 & 1/2 \end{pmatrix} \end{aligned} \tag{4}$$

or

$$B \equiv \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \tag{5}$$

2. This is an example of two incompatible measurements. The probability of measuring a_1 after the b measurement is

$$p(a_1) = \sum_r p(a_1|b_r)p(b_r) \tag{6}$$

After the a_1 measurement, the state is given by

$$|a_1\rangle = \frac{2}{\sqrt{13}}|b_1\rangle + \frac{3}{\sqrt{13}}|b_1\rangle \tag{7}$$

so the probabilities of measuring b_1 and b_2 are

$$p(b_1) = \frac{4}{13}, \qquad p(b_2) = \frac{9}{13} \tag{8}$$

Now

$$\begin{pmatrix} |b_1\rangle \\ |b_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} |a_1\rangle \\ |a_2\rangle \end{pmatrix} \tag{9}$$

so that

$$p(a_1|b_1) = \frac{4}{13}, \qquad p(a_1|b_2) = \frac{9}{13} \tag{10}$$

The probability of obtaining a_1 on the second measurement is then

$$p(a_1) = \sum_r p(a_1|b_r)p(b_r) = \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2 = \frac{97}{169}. \tag{11}$$

3. (a) In this normalized state, $\langle x\rangle = 0$, $\langle x^2\rangle = \Delta^2$, and

$$\langle p\rangle = \int dx\psi^*(x)\Big(-i\hbar\frac{\partial}{\partial x}\Big)\psi(x) = \int dx|\psi(x)|^2\frac{-i\hbar x}{2\Delta^2} = 0 \tag{12}$$

and

$$\Delta p^2 = \langle p^2\rangle = \int dx\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi(x) = \hbar^2\int dx|\psi(x)|^2[\frac{1}{2\Delta^2} - \frac{x^2}{4\Delta^2}] = \frac{\hbar^2}{4\Delta^2} \tag{13}$$

so that $\Delta p = \hbar/2\Delta$.

(b) The correctly normalized state is $\psi(x) = \frac{1}{\sqrt{2a}}\theta(a-|x|)$. In momentum space,

$$\phi(p) = \int_{-a}^{a} dx\frac{1}{\sqrt{4\pi\hbar a}}e^{-i\frac{px}{\hbar}} = \sqrt{\frac{a}{\pi\hbar}}\frac{sin(\frac{pa}{\hbar})}{\frac{pa}{\hbar}} \tag{14}$$

So that since $|\phi(p)|^2$ is even in momentum,

$$\begin{aligned} \langle p\rangle &= \int dp|\phi(p)|^2 p = 0, \\ \langle p^2\rangle &= \int dpp^2|\phi(p)|^2 = \frac{\hbar}{a\pi}\int \sin^2(\frac{pa}{\hbar}) = \infty. \end{aligned} \tag{15}$$

(c) If you chose the normalization $\langle x|p\rangle = e^{ipx/\hbar}$ then $|p\rangle = \sqrt{2\pi\hbar}|\tilde{p}\rangle$, where $|\tilde{p}\rangle$ is normalized so that $\langle\tilde{p}|\tilde{p}'\rangle = \delta(\tilde{p}-\tilde{p}')$. From the completeness relation of $|\tilde{p}\rangle$, we have

$$1 = \int d\tilde{p}|\tilde{p}\rangle\langle\tilde{p}| = \int \frac{dp}{2\pi\hbar}|p\rangle\langle p| \tag{16}$$

(d) The expectation value of the position is given by

$$\langle\psi|\hat{x}|\psi\rangle = \int \psi^*(p)\left(i\hbar\frac{\partial}{\partial p}\right)\psi(p) \tag{17}$$

4. The ground-state energy is given by

$$E = \langle\frac{\Delta p^2}{2m} - \frac{k}{r^{3/2}}\rangle \geq \frac{\Delta p^2}{2m} - \frac{k}{\Delta r^{3/2}} \tag{18}$$

Now writing $\Delta x = \Delta y = \Delta z = \Delta r/\sqrt{3}$, and $\Delta p_x = \Delta p_y = \Delta p_z = \Delta p/\sqrt{3}$,

$$\Delta p \Delta r \geq \frac{3\hbar}{2} \tag{19}$$

We may then write

$$E \geq \frac{1}{m}\left(\frac{\Delta p^2}{2} - \frac{2}{3}\Delta p^{3/2}\sqrt{\kappa}\right) \tag{20}$$

where

$$\kappa = \frac{k^2 m^2}{(\frac{3}{2})^{\frac{1}{2}}\hbar^3} \tag{21}$$

is a characteristic momentum. Minimizing w.r.t. Δp gives $\Delta p = \kappa$, so that

$$E = -\frac{\kappa^2}{6m} = -\frac{1}{9}\frac{(km)^4}{\hbar^6}. \tag{22}$$

(I am basically happy if you are able to get the right combination of constants in E, and the precise multiplying factor will depend on details of your approximations.