GRADUATE QUANTUM MECHANICS: 501 Fall 2001

Solution to assignment I.

1. (a) We expand the trace using a complete set of states and rearrange the terms to obtain:

$$\begin{aligned}\mathrm{Tr}(XY) &= \sum_a \langle a|XY|a\rangle = \sum_{a,b}\langle a|X\overbrace{|b\rangle\langle b|}^{1}Y|a\rangle \\ &= \sum_{a,b}\langle b|Y\overbrace{|a\rangle\langle a|}^{1}X|b\rangle = \sum_b \langle b|YX|b\rangle = \mathrm{Tr}(YX)\end{aligned} \tag{1}$$

(b) Between any two states $|a\rangle$ and $|b\rangle$, we have

$$\langle a|(XY)|b\rangle^* = \langle b|(XY)^\dagger|a\rangle \tag{2}$$

We can also write this as

$$[(\langle a|X)\,(Y|b\rangle)]^* = \left(\langle b|Y^\dagger\right)\left(X^\dagger|a\rangle\right) = \langle b|Y^\dagger X^\dagger|a\rangle \tag{3}$$

Comparing the two expressions, we deduce that $(XY)^\dagger = Y^\dagger X^\dagger$.

(c) In the eigenket basis, for any function $G(\hat{A})$ that can be written as a Taylor series, if $\hat{A}\langle a| = a\langle a|$ then $G[\hat{A}]|a\rangle = G[a]|a\rangle$. Thus since the basis is complete,

$$\exp[if(\hat{A})] = \sum_a \exp[if(\hat{A})]|a\rangle\langle a| = \sum_a \exp[if(\hat{a})]|a\rangle\langle a| \tag{4}$$

(d) By substituting $\psi_a(x) = \langle x|a\rangle$, $\psi_a^*(y) = \langle a|y\rangle$we have

$$\sum_a \psi_a^*(y)\psi_a(x) = \sum_a \langle a|y\rangle\langle x|a\rangle = \sum_a \langle x|a\rangle\langle a|y\rangle = \langle x|y\rangle = \delta^{(3)}(x-y) \tag{5}$$

2. Since the states $|1\rangle$ and $|2\rangle$ are the eigenkets of some observable, they provide an orthonormal basis. In this basis, the Hamiltonian becomes

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \tag{6}$$

Note that since H must be Hermitian, this means that h_{12} is real. If $|\psi\rangle = \sum_{a=1,2}\psi_a|a\rangle$ is an eigenket of H, then

$$\sum H_{ab}\psi_b = E\psi_a \tag{7}$$

We can then go ahead and derive the eigenvalues from the determinantly equation $\det(H - E\mathbf{1}) = 0$, back-substituting into the above expression to derive the eigenvectors ψ_a. An alternative approach is to write H in the form

$$H = a + \vec{b}\cdot\vec{\sigma} \tag{8}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Since $\mathrm{Tr}[\sigma_a \sigma_b] = 2\delta_{ab}$, and $\mathrm{Tr}\sigma_a = 0$, we can obtain

$$a = \frac{1}{2}\mathrm{Tr}[H\mathbf{1}] = \frac{1}{2}(h_{11} + h_{22}) \tag{9}$$

and

$$\vec{b} = \frac{1}{2}\mathrm{Tr}[H\vec{\sigma}] = (h_{12}, 0, \frac{1}{2}(h_{11} - h_{22})) \tag{10}$$

We can write $\vec{b} = b\hat{n}$ where $\hat{n}$ is the unit vector $\hat{n} = \vec{b}/|b|$ and $b \equiv |b|$. We can then write

$$H = a + b\hat{n} \cdot \vec{\sigma}. \tag{11}$$

Written in the above form, it is easy to see that the eigenvalues of H are

$$E = a \pm b = \frac{1}{2}(h_{11} + h_{22}) \pm \sqrt{[\frac{1}{2}(h_{11} - h_{22})]^2 + h_{12}^2} \tag{12}$$

corresponding to the two eigenkets of the spin operator $\hat{n} \cdot \vec{\sigma}$,

$$(\hat{n} \cdot \vec{\sigma})|\hat{n}; \pm\rangle = \pm|\hat{n}; \pm\rangle \tag{13}$$

From the information in the question, these two states are given by

$$\begin{aligned} |\mathbf{n}; +\rangle &= \cos\frac{\beta}{2}|1\rangle \pm e^{i\alpha}\sin\frac{\beta}{2}|2\rangle \\ |\mathbf{n}; -\rangle &= -\sin\frac{\beta}{2}|1\rangle \pm e^{i\alpha}\cos\frac{\beta}{2}|2\rangle \end{aligned} \tag{14}$$

where α and β are the polar co-ordinates of the unit vector $\hat{n} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$. The question gave the result for the upper state, we have chosen the lower state $|\hat{n}; -\rangle$ to be orthogonal to this $|\hat{n}; +\rangle$. Since $n_y = 0$, we deduce that $\alpha = 0$. We also have $\cos\beta = b_z/b$, from which we deduce that

$$\begin{aligned} \cos\frac{\beta}{2} &= \left[\frac{1 + \cos\beta}{2}\right]^{\frac{1}{2}} = \left[\frac{1 + b_z/b}{2}\right]^{\frac{1}{2}} \\ \sin\frac{\beta}{2} &= \left[\frac{1 - \cos\beta}{2}\right]^{\frac{1}{2}} = \left[\frac{1 - b_z/b}{2}\right]^{\frac{1}{2}} \end{aligned} \tag{15}$$

The final expression for the two eigenkets is then

$$\begin{aligned} |+\rangle &= \left\{\frac{1}{2}\left[1 + \frac{(h_{11} - h_{22})}{[(h_{11} - h_{22})^2 + (2h_{12})^2]^{\frac{1}{2}}}\right]\right\}^{\frac{1}{2}}|1\rangle + \left\{\frac{1}{2}\left[1 - \frac{(h_{11} - h_{22})}{[(h_{11} - h_{22})^2 + (2h_{12})^2]^{\frac{1}{2}}}\right]\right\}^{\frac{1}{2}}|2\rangle \\ |-\rangle &= \left\{\frac{1}{2}\left[1 - \frac{(h_{11} - h_{22})}{[(h_{11} - h_{22})^2 + (2h_{12})^2]^{\frac{1}{2}}}\right]\right\}^{\frac{1}{2}}|1\rangle - \left\{\frac{1}{2}\left[1 + \frac{(h_{11} - h_{22})}{[(h_{11} - h_{22})^2 + (2h_{12})^2]^{\frac{1}{2}}}\right]\right\}^{\frac{1}{2}}|2\rangle \end{aligned}$$

3. First let us expand

$$F(p) = \sum_{n=0,\infty} f_n p^n \tag{16}$$

as a Taylor series. To find the commutator with q, we need to know

$$[q, F(p)] = \sum_{n=0,\infty} f_n [q, p^n] \tag{17}$$

Now using the result $[A, BC] = B[A, C] + [A, B]C$, we may deduce that

$$\begin{aligned} [q, p^2] &= p[q, p] + [q, p]p = 2(i\hbar)p, \\ [q, p^3] &= p[q, p^2] + [q, p]p^2 = 3(i\hbar)p^2 \end{aligned} \tag{18}$$

and by induction,

$$[q, p^n] = i\hbar n p^{n-1} \tag{19}$$

so that

$$[q, F(p)] = i\hbar \sum_{n=0,\infty} f_n n p^{n-1} == i\hbar F'[\hat{p}] \tag{20}$$

where we have made the crucial identification, $\sum_{n=0,\infty} f_n n p^{n-1} = F'(p)$, where $F'(p)$ is the first derivative of the function $F(p)$.

4. The short answer is no! In order to make sure that the spurious fields B_1 are small compared with the dipole fields, one must make sure that the momentum of the electron is zero, within a certain tolerance Δp. But to measure the field B_2, one needs to make sure that the position r is known sufficiently accurately so that the uncertainty in B_2 is much smaller than B_2. These two requirements imply that $\Delta x \Delta p << \hbar/2$. But the uncertainty relation implies the exact opposite, and for this reason, the measurement is not possible on a free electron.

   To see this more explicitly, note that $B_1 << B_2$ implies

   $$evr << \mu_e \tag{21}$$

   But since $\Delta v << v$ and $\Delta x << r$, this implies, $\Delta v \Delta x << (\mu_e/e)$ or $\Delta x \Delta p << (m\mu_e/e) = \hbar/2$, where we have substituted $\mu_e = e\hbar/2m$. This directly contradicts the uncertainty principle $\Delta x \Delta p \geq \hbar/2$.