Lecture 3

SU(2)

January 26, 2011
A group is a set of elements plus a composition rule, such that:

1. Combining two elements under the rule gives another element of the group.
   \[ E \cdot E' = E'' \]

2. There is an identity element
   \[ E \cdot I = I \cdot E = E \]

3. Every element has a unique inverse
   \[ E \cdot E^{-1} = E^{-1} \cdot E = I \]

4. The composition rule is associative
   \[ A \cdot (B \cdot C) = (A \cdot B) \cdot C \]
The $U(1)$ Group

The set of all functions $U(\theta) = e^{i\theta}$ form a group.

$$E(\theta) \cdot U(\theta') = e^{i(\theta + \theta')} = E(\theta + \theta')$$

$I = E(0)$

$E^{-1}(\theta) = E(-\theta)$

$$(E(\theta_1) \cdot E(\theta_2)) \cdot U(\theta_3) = E(\theta_1) \cdot (E(\theta_2) \cdot E(\theta_3))$$

This is the one dimensional unitary group

$U(1)$
Lie Groups

If the elements of a group are differentiable with respect to their parameters, the group is a Lie group.

$U(1)$ is a Lie group.

\[
\frac{dE}{d\theta} = iE
\]

For a Lie group, any element can be written in the form

\[
E(\theta_1, \theta_2, \ldots, \theta_n) = \exp \left( \sum_{i=1}^{n} i\theta_i F_i \right)
\]

The quantities $F_i$ are the generators of the group.

The quantities $\theta_i$ are the parameters of the group. They are a set of $i$ real numbers that are needed to specify a particular element of the group.

Note that the number of generators and parameters are the same. There is one generator for each parameter.
The group $U(1)$ is the set of all one dimensional, complex unitary matrices.

The group has one generator $F = 1$, and one parameter, $\theta$.

It simply produces a complex phase change.

$$E(\theta) = e^{-i\theta F} = e^{-i\theta}$$

Since the generator $F$, commutes with itself, the group elements also commute.

$$E(\theta_1)E(\theta_2) = e^{-i\theta_1}e^{-i\theta_2} = e^{-i\theta_2}e^{-i\theta_1} = E(\theta_2)E(\theta_1)$$

Such groups are called Abelian groups.
The group $U(2)$ is the set of all two dimensional, complex unitary matrices.

An complex $n \times n$ matrix has $2n^2$ real parameters. The unitary condition constraint removes $n^2$ of these.

The group $U(2)$ then has four generators and four parameters.

$$E(\theta_0, \theta_1, \theta_2, \theta_3) = e^{-i\theta_j F_j} \quad \text{where} \quad j = 0, 1, 2, 3$$

The generators are: $F_i = \sigma_i / 2$

$$F_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad F_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$F_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad F_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
SU(2)

The operators represented by the elements of $U(2)$ act on two dimensional complex vectors.

The operations generated by $F_0 = \sigma_0/2$ simply change the complex phase of both components of the vector by the same amount. In general we are not so interested in these operations.

The group $SU(2)$ is the set of all two dimensional, complex unitary matrices with unit determinant.

The unit determinant constraint removes one more parameter. The group $SU(2)$ then has three generators and three parameters.

$$E(\theta_1, \theta_2, \theta_3) = e^{-i\theta_j F_j} \quad \text{where} \quad j = 1, 2, 3$$

The generators of $SU(2)$ are a set of three linearly independent, traceless $2 \times 2$ Hermitian matrices:

$$F_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad F_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad F_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since the generators do not commute with one another, this is a non Abelian group.
The group $SO(3)$ is the set of all three dimensional, real orthogonal matrices with unit determinant. [Requiring that the determinant equal 1 and not $-1$, excludes reflections.]

A real $n \times n$ orthogonal matrix has $n(n-1)/2$ real parameters

The group $SO(3)$ then has three parameters.

The group $SO(3)$ represents the set of all possible rotations of a three dimensional real vector.

For example, in terms of the Euler angles

$$R(\alpha, \beta, \gamma) =$$

$$\begin{pmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}$$
SU(2) and SO(3)

Both the groups $SU(2)$ and $SO(3)$ have three real parameters.

For example the $SU(2)$ elements can be parametrized by:

$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$|a|^2 + |b|^2 = 1$

$\begin{pmatrix} \cos \theta e^{i\alpha} & \sin \theta e^{i\gamma} \\ -\sin \theta e^{-i\gamma} & \cos \theta e^{-i\alpha} \end{pmatrix}$

In fact, there is a one-to-one correspondence between $SU(2)$ and $SO(3)$ such that $SU(2)$ represents the set of all possible rotations of two dimensional complex vectors (spinors) in a real three dimensional space.

Well not quite. There are actually two $SU(2)$ rotations for every $SO(3)$ rotation. That’s because, for $SU(2)$, the rotation with $\theta_i + 2\pi$ is not the same as the rotation with $\theta_i$. There is a sign difference. For $SU(2)$, the range of $\theta_i$ is: $0 \leq \theta_i \leq 4\pi$. The set with $0 \leq \theta_i \leq 2\pi$ corresponds to the complete set of $SO(3)$ rotations.
The Group SU(2)

As we have seen, for the case of $j = 1/2$, rotations are represented by the matrices of the form

$$e^{-i\theta \vec{\sigma} \cdot \hat{\theta} / 2} = \cos(\theta/2)I - i(\vec{\sigma} \cdot \hat{\theta}) \sin(\theta/2)$$

These matrices are $2 \times 2$ complex unitary matrices with unit determinant. The determinant is unity because

$$\det(e^{-i\theta \vec{\sigma} \cdot \hat{\theta} / 2}) = e^{Tr(-i\theta \vec{\sigma} \cdot \hat{\theta} / 2)} = e^0 = 1$$

The set of all of these matrices forms the group SU(2) under the operation of matrix multiplication.

These elements are described by a set of three real parameter $(\theta_x, \theta_y, \theta_z)$

$$\vec{\sigma} \cdot \hat{\theta} = \begin{pmatrix} \hat{\theta}_z & \hat{\theta}_x - i\hat{\theta}_y \\ \hat{\theta}_x + i\hat{\theta}_y & -\hat{\theta}_z \end{pmatrix}$$

This set of matrices are then elements of a three dimension real vector space that can be identified as the space of physics vectors and the three generators of rotations, $\sigma_1, \sigma_2, \sigma_3$ can be identified with the three components of a physical vector

$$S_x = \frac{\hbar}{2} \sigma_1 \quad S_y = \frac{\hbar}{2} \sigma_2 \quad S_z = \frac{\hbar}{2} \sigma_3$$
SU(2) and Rotations

Any normalized element of a complex two dimensional vector space \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \) is also described by three real parameters, the real and imaginary parts of \( \alpha \) and \( \beta \) with the constraint \( |\alpha|^2 + |\beta|^2 = 1 \)

Any normalized element of the two dimensional vector space can be obtained by a rotation of the state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

\[
e^{-i \vec{\sigma} \cdot \hat{n} / 2} = \cos(\theta/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin(\theta/2) \begin{pmatrix} \hat{\theta}_z & \hat{\theta}_x - i \hat{\theta}_y \\ \hat{\theta}_x + i \hat{\theta}_y & -\hat{\theta}_z \end{pmatrix}
\]

\[
= \begin{pmatrix} \cos \frac{\theta}{2} - i \hat{\theta}_z \sin \frac{\theta}{2} & (-\hat{\theta}_y - i \hat{\theta}_x) \sin \frac{\theta}{2} \\ (\hat{\theta}_y - i \hat{\theta}_x) \sin \frac{\theta}{2} & \cos \frac{\theta}{2} + i \hat{\theta}_z \sin \frac{\theta}{2} \end{pmatrix}
\]

\[
e^{-i \vec{\sigma} \cdot \hat{n} / 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} - i \hat{\theta}_z \sin \frac{\theta}{2} \\ (\hat{\theta}_y - i \hat{\theta}_x) \sin \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]

Conversely, for every state \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \) there is some direction \( \hat{n} \) such that:

\[
\vec{S} \cdot \hat{n} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hat{n}}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]
A Check on Consistency

Under a $-\pi/2$ rotation about the $x$-axis, the expectation value of $\hat{S}_y$ for the rotated system should equal the expectation value of $\hat{S}_z$ for the non-rotated system.

$$\langle \psi' | \hat{S}_y | \psi' \rangle = \langle \psi | \hat{S}_z | \psi \rangle$$

Let’s check it.

$$\langle \psi | \hat{S}_z | \psi \rangle = \langle \psi | e^{-i\pi \sigma_x/4} e^{i\pi \sigma_x/4} \hat{S}_z e^{-i\pi \sigma_x/4} e^{i\pi \sigma_x/4} | \psi \rangle$$

$$= \langle \psi' | e^{i\pi \sigma_x/4} \hat{S}_z e^{-i\pi \sigma_x/4} | \psi' \rangle$$

Now, does

$$e^{i\pi \sigma_x/4} \hat{S}_z e^{-i\pi \sigma_x/4} = \hat{S}_y$$
Generalization of the Anticommutation Relations

\[ \sigma_i \sigma_j = -\sigma_j \sigma_i \quad \text{for} \quad i \neq j \]

\[ \Rightarrow \quad \sigma_i \sigma_j^n = (-1)^n \sigma_j^n \sigma_i = (-\sigma_j)^n \sigma_i \]

Then for any analytic function of \( \sigma_j \), i.e. and function that can be expanded as a power series, we have

\[ \sigma_i f(\sigma_j) = f(-\sigma_j) \sigma_i \]

Using this result we have

\[ e^{i \pi \sigma_x/4} \hat{S}_x e^{-i \pi \sigma_x/4} = \frac{\hbar}{2} e^{i \pi \sigma_x/4} \sigma_z e^{-i \pi \sigma_x/4} \]

\[ = \frac{\hbar}{2} e^{i \pi \sigma_x/4} e^{i \pi \sigma_x/4} \sigma_z = \frac{\hbar}{2} e^{i \pi \sigma_x/2} \sigma_z \]

\[ = \frac{\hbar}{2} (\cos \frac{\pi}{2} + i \sigma_x \sin \frac{\pi}{2}) \sigma_z = \frac{\hbar}{2} i \sigma_x \sigma_z = \frac{\hbar}{2} \sigma_y = \hat{S}_y \]

It checks.
3-Dimensional Representation of $SU(2)$

The **structure** of a group is defined by the algebra of its generators. For $SU(2)$ this is:

$$[F_i, F_j] = i\epsilon_{ijk}F_k$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}$$

We can find a set of three $3 \times 3$ complex, traceless, Hermitian matrices that satisfy this same algebra.

$$F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$F_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

These generate the three dimensional representation of $SU(2)$.

Note that they do not represent all possible rotations of a three dimensional, normalized complex vector. That would require at least five parameters. For example in the homework you show that you cannot rotate the state $|jm\rangle = |1,1\rangle$ into $|1,0\rangle$. 


We are used to $SU(2)$ in spin space
- Transformations are physical rotations in 3-dimensional space
- Generators are angular momentum operators

Now we want to consider $SU(2)$ in a new isospin space.

Complete analogy with $SU(2)$ in spin space.

But, transformations are not physics rotations but rather rotations in the abstract isospin space.

$$\psi = e^{-i k \cdot x} \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} \begin{pmatrix} c_u \\ c_d \end{pmatrix}$$
The group $SU(3)$ is the set of all three dimensional, complex unitary matrices with unit determinant.

This set has $2(3)^2 - (3)^2 - 1 = 8$ parameters and generators.

$$E(\theta_1, \theta_2, \cdots, \theta_8) = e^{-i\theta_j F_j} \quad \text{where} \quad j = 1, 2, \cdots, 8$$

The generators of $SU(3)$ are a set of eight linearly independent, traceless $3 \times 3$ Hermitian matrices:

Since there are eight generators, the $SU(3)$ elements represent rotations of complex three component vectors in an eight dimensional space.
The structure of $SU(3)$ is:

$$[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c$$

$$f_{123} = 1 \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

$$f_{147} = f_{516} = f_{246} = f_{257} = f_{345} = f_{637} = \frac{1}{2}$$

$$f_{abc} \quad \text{is totally antisymmetric}$$