Lecture 15

Two Dimensional Isotropic Harmonic Oscillator

October 27, 2010
2-Dimensional Isotropic Oscillator

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} \right) + \frac{1}{2} m \omega^2 x^2 + \frac{1}{2} m \omega^2 y^2 - E \right] u_E(x, y) = 0
\]

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right) u_E(x, y) + \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m \omega^2 y^2 \right) u_E(x, y) = Eu_E(x, y)
\]

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right) u_E(x, y) = E_x u_E(x, y)
\]

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m \omega^2 y^2 \right) u_E(x, y) = E_y u_E(x, y)
\]

\[
E = E_x + E_y
\]

\[
E_x = \hbar \omega (n_x + 1/2) \quad E_y = \hbar \omega (n_y + 1/2)
\]

\[
E = \hbar \omega (n_x + n_y + 1) = \hbar \omega (n + 1)
\]

\[
u_n(x, y) = u_{n_x}(x) u_{n_y}(y)
\]
Degeneracy

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n_x$</th>
<th>$n_y$</th>
<th>degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

degeneracy $= n + 1$

There must be some symmetry that is causing different states to have the same energy.

There must be another Hermitian operator that commutes with the Hamiltonian and whose eigenvalues can be used to distinguish the degenerate eigenstates.
Degeneracy & Symmetry

If $V(x)$ is independent of $x$, then there is translational symmetry. As a result

$$\hat{E} \hat{p} u(x) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \right) \left( -\hbar \frac{\partial}{\partial x} \right) u(x)$$

$$= \left( -\hbar \frac{\partial}{\partial x} \right) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \right) u(x) = \hat{p} \hat{E} u(x)$$

This means that $u(x)$ can simultaneously be an eigenstate of both $\hat{p}$ and $\hat{E}$. Then, there is degeneracy since for a given energy eigenstate two momentum eigenstates are allowed.

$$e^{ipx/\hbar} \quad \text{and} \quad e^{-ipx/\hbar}$$

This is not true if $V(x)$ depends on $x$, e.g., a well potential, since

$$x \frac{\partial}{\partial x} u(x) \neq \frac{\partial}{\partial x} [xu(x)] \quad \Rightarrow \quad \hat{E} \hat{p} u(x) \neq \hat{p} \hat{E} u(x)$$

and $u(x)$ cannot be simultaneously an eigenstate of $\hat{E}$ and $\hat{p}$.

$$e^{ipx/\hbar} + e^{-ipx/\hbar} \quad \text{and} \quad e^{ipx/\hbar} - e^{-ipx/\hbar}$$

have different energies.