Second Postulate of Quantum Mechanics

Postulate 2

For every observable, there corresponds a linear Hermitian operator that acts on the vector state of the system.

Examples of observables:

- momentum of an electron
- position of an electron
- polarization of a photon
An operator $\Omega$ transforms a vector $|V\rangle$ into another vector $|V'\rangle$ in the same vector space.

$$\Omega |V\rangle = |V'\rangle$$

A linear operator satisfies the following condition:

$$\Omega (\alpha |V\rangle + \beta |W\rangle) = \alpha \Omega |V\rangle + \beta \Omega |W\rangle$$

Once the action of a linear operator on the basis states is known, its action on any vector in the space is determined.

$$\Omega |i\rangle = |i'\rangle$$

$$\Rightarrow \quad \Omega |V\rangle = \sum_i \Omega v_i |i\rangle = \sum_i v_i \Omega |i\rangle = \sum_i v_i |i'\rangle$$
Some Examples of Linear Operators

- Identity operator: $I$
  $$I |V\rangle = |V\rangle$$

- Projection operator of the $i'$th basis vector: $P_i$
  $$P_i |V\rangle = v_i |i\rangle$$

- Rotation of a three dimensional arrow by $90^\circ$ about the $|1\rangle$ axis: $R_1(\pi/2)$
  $$R_1(\pi/2) |1\rangle = |1\rangle$$
  $$R_1(\pi/2) |2\rangle = |3\rangle$$
  $$R_1(\pi/2) |3\rangle = -|2\rangle$$
Matrix Representation of Operators

In a basis independent way, we write:

\[ |V'\rangle = \Omega |V\rangle \]

In a specific basis, just as we can represent \(|V\rangle\) and \(|V'\rangle\) as \(1 \times n\) column matrices, we can represent the operator \(\Omega\) as an \(n \times n\) matrix.

\[
\begin{pmatrix}
v'_1 \\
v'_2 \\
\vdots \\
v'_n
\end{pmatrix} =
\begin{pmatrix}
\Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\
\Omega_{21} & \Omega_{22} & \cdots & \Omega_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\Omega_{n1} & \Omega_{1n2} & \cdots & \Omega_{nn}
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix}
\]

where \(\Omega_{ij} = \langle i | \Omega | j \rangle\)
Examples of Matrix Representations

Identity

\[
I = \begin{pmatrix}
1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & \ldots & \ldots & \\
\ldots & \ldots & 1 & \ldots & \\
\ldots & \ldots & \ldots & 1 & 0 \\
0 & \ldots & \ldots & 0 & 1
\end{pmatrix}
\]

Projection

\[
P_i = \begin{pmatrix}
0 & \ldots & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & 1 & \ldots & \\
\ldots & \ldots & \ldots & 0 & \ldots \\
0 & \ldots & \ldots & 0 & 0
\end{pmatrix}
\]

Rotation

\[
R_1(\pi/2) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\]
Completeness Relation

Note that we can write $P_i$ as

$$P_i = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix} (0 \cdots 0 1 0 \cdots 0) = |i\rangle\langle i|$$

We can then write the identity operator as

$$I = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} |i\rangle\langle i|$$
Consider an operator that is the product of two operators

\[ |V'\rangle = \Omega \Lambda |V\rangle \]

In a given basis, the operation corresponding to the product is given by the product of the respective matrices

\[
(\Omega \Lambda)_{ij} = \langle i | \Omega \Lambda | j \rangle = \langle i | \Omega I \Lambda | j \rangle = \sum_k \langle i | \Omega | k \rangle \langle k | \Lambda | j \rangle = \sum_k \Omega_{ik} \Lambda_{kj}
\]

This is the definition of multiplication of the two matrices \( \Omega \) and \( \Lambda \)
Unlike numbers, matrices and therefore operators in general do not commute with each other.

The order in which the operations are applied matters.

In general

$$\Omega \Lambda |V\rangle \neq \Lambda \Omega |V\rangle$$

As you may know, this has important consequences.

We define the commutation of $\Omega$ and $\Lambda$ as

$$[\Omega, \Lambda] \equiv \Omega \Lambda - \Lambda \Omega$$
Adjoint Operator

Corresponding to

$$|V'\rangle = \Omega |V\rangle$$

we have

$$\langle V | \Omega^\dagger = \langle V'|$$

where $\Omega^\dagger$ is the adjoint operator to $\Omega$.

$$\langle i | \Omega^\dagger | j \rangle = \langle i' | j \rangle = \langle j | i' \rangle^* = \langle j | \Omega | i \rangle^*$$

$$\Rightarrow \quad \Omega^\dagger_{ij} = \Omega^*_{ji}$$

$\Omega^\dagger$ is the complex-conjugate transpose of $\Omega$
also known as the Hermitian conjugate of $\Omega$
We will be essentially only be interested in operators that are either Hermitian or Unitary.

Hermitian: \( H^\dagger = H \)

Unitary: \( U^\dagger = U^{-1} \)

\( UU^\dagger = U^\dagger U = I \)

The inner product is invariant under unitary operations.

if \( |V'_1\rangle = U|V_1\rangle \) and \( |V'_2\rangle = U|V_2\rangle \)

\[ \langle V'_2|V'_1\rangle = \langle V_2|U^\dagger U|V_1\rangle = \langle V_2|V_1\rangle \]

Unitary operators are a generalization of orthogonal transformations (rotations) to a complex vector space.
Eigenvalues and Eigenvectors

In general, an operator will transform one vector into another

$$\Omega |V\rangle = |V'\rangle$$

However, for a given operator, there will be some vectors for which the operation is simply a multiplication of the vector by a number.

$$\Omega |V\rangle = \omega |V\rangle$$

These vectors are the eigenvectors of the operator and the numbers are the corresponding eigenvalues.

You may know that eigenvectors and eigenvalues play a very special role in quantum mechanics.
Eigenvalues of Hermitian Operators

The eigenvalues of Hermitian operators are real numbers.

\[
H|V\rangle = |V'\rangle = h|V\rangle \quad \Rightarrow \quad \langle V|H|V\rangle = h\langle V|V\rangle
\]

\[
\langle V|H^\dagger|V\rangle = \langle V|H|V\rangle \quad \Rightarrow \quad h^*\langle V|V\rangle = h\langle V|V\rangle
\]

\[
\Rightarrow \quad h^* = h
\]
The eigenvalues of unitary operators are complex numbers of unit modulus.

\[ U |V \rangle = |V' \rangle = u |V \rangle \]

\[ \langle V | U^\dagger = \langle V' | = \langle V | u^* \]

\[ \langle V | V \rangle = \langle V | U^{-1} U | V \rangle = \langle V | U^\dagger U | V \rangle = u^* u \langle V | V \rangle \]

\[ \Rightarrow \quad u^* u = 1 \]